On Some Aspects of Decidability of Annotated Systems

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Abstract *In this paper, we discuss some results of decidability of annotated systems. The annotated propositional logic* $\mathbf{P}\tau$ *and its S5 type modal extension* $\mathbf{S5}\tau$ *are shown to be decidable. These results reveal that annotated systems are computationally attractive as the foundations of paraconsistent reasoning.*

Keywords: annotated logics, paraconsistent logics, decidability.

1 Introduction

Annotated logics, as proposed by Subrahmanian [11], are a kind of paraconsistent and in general paracomplete and non-alethic logics. These systems were originally designed to give the foundations of reasoning about inconsistency. Another motivation in studying annotated systems lies in some applications. Annotated logics can be applied to many themes in AI, Robotics and the theory of electronic circuit.

This paper discusses some results concerning decidability of annotated systems. The annotated propositional logic $\mathbf{P}\tau$ and its S5 type modal extension $\mathbf{S5}\tau$ are shown to be decidable. These results reveal that annotated systems are computationally attractive as the foundations of paraconsistent reasoning.

2 Annotated Propositional Logics $\mathbf{P}\tau$

In this section, we introduce formally the paraconsistent annotated propositional logics $\mathbf{P}\tau$. A detailed account is to be found in da Costa, Subrahmanian and Vago[9], da Costa, Abe and Subrahmanian [8], Abe [1] and Abe and Akama [3]. In this paper, we assume that if S is a set then $\sharp S$ indicates the cardinality of S. Other usual terminology of naive set theory will be employed without major comments.

The symbol $\tau = \langle | \tau |, \leq, \sim \rangle$ denotes some fixed finite lattice called *lattice of truth-values*. We use the symbol $\leq$ to denote the ordering in which τ is a complete lattice, $\perp$ and $\top$ to denote, respectively, the bottom element and top element of τ. Also, $\wedge$ and $\vee$ indicate, respectively, the greatest lower bound and the least upper bound operators with respect to subsets of $|\tau|$. We also fix an operator $\sim : |\tau| \rightarrow |\tau|$ which will work as the "meaning" of the negation of the system $\mathbf{P}\tau$. The language of $\mathbf{P}\tau$ has the following (denumerable) primitive symbols.

1. Propositional symbols: $p, q, r, ...$

2. Connectives, i.e. $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication).

3. Each member of τ is called an *annotated constant*: $\lambda, \mu, \theta, ...$

4. Auxiliary symbols: (,).

Formulas are now defined as follows:

1. If p is a propositional symbol and $\lambda \in \tau$ is an annotated constant, then p_λ is an atomic formula.
2. A and B are formulas, then $\neg A, A \wedge B, A \vee B, A \rightarrow B$ are formulas.
3. Only those expressions are formulas that are determined to be so by means of conditions 1 and 2.

The atomic formula p_λ can be read "it is believed that p's truth-value is at least λ". Let A be a formula. Then, $\neg^0 A$ is A, $\neg^1 A$ is $\neg A$, and $\neg^k A$ is $\neg(\neg^{k-1} A)$, where $k \geq 0$ is the natural number. The convention is also used for $\sim$. If p is a propositional symbol and λ is an annotated constant, then the formula of the form $\neg^k p_\lambda$ is called a *hyper-literal.* A formula which is not a hyper-literal is called a *complex formula.*

Definition 2.1

Let A and B be formulas. Then we put
$A \leftrightarrow B =_{def} (A \rightarrow B) \wedge (B \rightarrow A)$
and $\neg_* A =_{def} A \rightarrow (A \rightarrow A) \wedge \neg(A \rightarrow A)$.

The formula $A \leftrightarrow B$ is read, as usual, the *equivalence* of A and B. The operator $\neg_*$ is called *strong negation*, so $\neg_* A$ must be read the *strong negation* of A.

The postulates (axiom schemata and primitive rules of inference) of $\mathbf{P}\tau$ are the following: A, B, and C are formulas whatsoever, F and G are complex formulas, p is a propositional symbol, and λ, μ, λ_j are annotated constants.

(1) $A \rightarrow (B \rightarrow A)$
(2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
(3) $((A \rightarrow B) \rightarrow A) \rightarrow A$
(4) $A, A \rightarrow B/B$
(5) $A \wedge B \rightarrow A$
(6) $A \wedge B \rightarrow B$
(7) $A \rightarrow (B \rightarrow (A \wedge B))$
(8) $A \rightarrow A \vee B$
(9) $B \rightarrow A \vee B$
(10) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$
(11) $(F \rightarrow G) \rightarrow ((F \rightarrow \neg G) \rightarrow \neg F)$
(12) $F \rightarrow (\neg F \rightarrow A)$
(13) $F \vee \neg F$
(14) $p_\perp$
(15) $\neg^k p_\lambda \rightarrow \neg^{k-1} p_{\sim\lambda}, k \geq 1$
(16) $p_\lambda \rightarrow p_\mu, \lambda \geq \mu$
(17) $p_{\lambda_1} \wedge p_{\lambda_2} \wedge ... \wedge p_{\lambda_m} \rightarrow p_\lambda$, where $\lambda = \bigvee_{i=1}^m \lambda_i$

Theorem 2.2

In $\mathbf{P}\tau$ all valid formulas of the classical positive propositional calculus are valid.

Theorem 2.3

In $\mathbf{P}\tau$, $\neg_*$ has all properties of the classical negation. For instrance, we have:

1. $\vdash A \vee \neg_* A$
2. $\vdash \neg_*(A \wedge \neg_* A)$
3. $\vdash (A \rightarrow B) \rightarrow ((A \rightarrow \neg_* B) \rightarrow \neg_* A)$
4. $\vdash A \rightarrow \neg_* \neg_* A$
5. $\vdash \neg_* A \rightarrow (A \rightarrow B)$
6. $\vdash A \rightarrow (\neg_* A \rightarrow B)$

Corollary 2.3.1

In $\mathbf{P}\tau$, the connectives $\neg_*, \wedge, \vee$, and $\rightarrow$ have all properties of the classical negation, conjunction, disjunction and implication, respectively.

Theorem 2.4

$\mathbf{P}\tau$ is non-trivial.

Proof: Let us consider the function $\phi : \mathbf{F} \rightarrow \mathbf{2}$ such that it associates each hyper-literal the value 1 and it extends to the remaining formulas according to the usual classical truth-tables. We say that A is pseudo-true if $\phi(A) = 1$ and pseudo-false if $\phi(A) = 0$. It is immediate to verify that all axioms are pseudo-true and modus ponens preserves pseudo-truth. However, if F is a complex formula, we have $\phi(F \wedge \neg F) = 0$.

Definition 2.5

Let Γ be a set of formulas The *syntactical consequence* of Γ, symbolized by $\bar{\Gamma}$, is the set $\bar{\Gamma} = \{A \in \mathbf{F} \mid \Gamma \vdash A\}$. If $\Gamma = \bar{\Gamma}$, then Γ is called a *theory.* A set of formulas Γ is called *trivial* if $\bar{\Gamma} = \mathbf{F}$. Otherwise Γ is *non-trivial.*

Theorem 2.6

Let Γ be a non-trivial set of formulas. Then Γ can be extended to a maximal (with respect to inclusion of sets) non-trivial set with respect to $\mathbf{F}$.

Proof: Let Γ be any non-trivial subset of formulas of $\mathbf{F}$. To show that Γ can be extended to a non-trivial maximal set, we construct a sequence $\Gamma_0, \Gamma_1, ..., \Gamma_n, ...$ as follows. As the vocabulary is composed by a denumerable set of symbols, the set of formulas of $\mathbf{F}$ is denumerable. Let $A_1, A_2, ..., A_n, ...$ be an enumeration of ther formulas of $\mathbf{F}$. Let $\Gamma_0 = \Gamma$, and inductively construct the rest of the sequence by taking $\Gamma_{i+1} = \Gamma \cup \{A_{i+1}\}$ if this set is non-trivial and otherwise by taking $\Gamma_{i+1} = \Gamma$. It is easy to see that each set of the sequence $\Gamma_0, \Gamma_1, ..., \Gamma_n, ..$ is non-trivial, and this is a non-decreasing sequence of sets such that $\Gamma \subseteq \Gamma_0 \subseteq \Gamma_1 \subseteq ... \subseteq \Gamma_n \subseteq ...$ We have that $\Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i$ is a non-trivial maximal set containing Γ. Each finite subset of Γ^* must be contained in some Γ_i for some i, and thus must be non-trivial (since Γ_i is non-trivial). It follows that Γ^* itself is non-trivial. We claim that in fact that Γ^* is a maximal non-trivial set. For suppose $A \in \mathbf{F}$ and $A \notin \Gamma^*$. As A is a formula of $\mathbf{F}$, it must appear in our enumeration, say as A_k. If $\Gamma \cup \{A_k\}$ were non-trivial, then our construction would guarantee that $A_k \in \Gamma_{k+1}$, and hence that $A_k \in \Gamma^*$. Because $A_k \notin \Gamma^*$, it follows that $\Gamma_k \cup \{A\}$ is also trivial. Hence $\Gamma^* \cup \{A\}$ is also trivial. It follows that Γ^* is a maximal non-trivial set.

As $\Gamma \subseteq \Gamma_i$, $i \in \omega$, it follows that $\Gamma \subseteq \Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i$. On the other hand, suppose that $\Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i$ is trivial. Thus, $\Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i = \mathbf{F}$. It follows that $p_\lambda, \neg_* p_\lambda \in \Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i$. As τ is finite, we have that any application of modus ponens has only a finite number of premises. Thus, there are $n, m < \omega$ such that $p_\lambda \in \Gamma_n$ and $\neg_* p_\lambda \in \Gamma_m$. Therefore $p_\lambda, \neg_* p_\lambda \in \Gamma_{n_0}$, where $n_0 = max(n, m)$. Thus Γ_{n_0} is trivial, which is a contradiction.

Theorem 2.7

Let Γ be a maximal non-trivial set of formulas. Then

1. If A is an axiom of $\mathbf{P}\tau$, then $A \in \Gamma$.
2. $A, B \in \Gamma$ iff $A \wedge B \in \Gamma$.
3. $A \vee B \in \Gamma$ iff $A \in \Gamma$ or $B \in \Gamma$.
4. If $p_\lambda, p_\mu \in \Gamma$, then $p_\theta \in \Gamma$. where $\theta = max(\lambda, \mu)$.
5. $\neg^k p_\mu \in \Gamma$ iff $\neg^{k-1} p_{\sim\mu} \in \Gamma$.
6. If $A, A \to B \in \Gamma$ then $B \in \Gamma$.
7. $A \to B \in \Gamma$ iff $A \notin \Gamma$ or $B \in \Gamma$.

Proof: Let us prove only 4. In fact, from $p_\lambda, p_\mu \in \Gamma$ it follows that $p_\lambda \wedge p_\mu \in \Gamma$ by 2. But it is an axiom of the form $p_\lambda \wedge p_\mu \to p_\theta$, where $\theta = max(\lambda, \mu)$. Thus, by 1 and 6 $p_\theta \in \Gamma$. The remaining cases are proved as in the classical cases.

Next, we describe a semantics for $\mathbf{P}\tau$; see Abe [1] for details. We denote by $\mathbf{P}$ the set of propositional symbols and by $\mathbf{A}$ the set of atomic formulas, and by $\mathbf{2}$ the set $\{0, 1\}$.

Definition 2.8

An *interpretation* (or $\mathbf{P}\tau$*-interpretation*) is a function $I : \mathbf{P} \to |\tau|$. For an interpretation I we can associate a valuation $V_I : \mathbf{F} \to \mathbf{2}$, inductively defined by:

1. If $p \in \mathbf{P}$ and $\mu \in |\tau|$, then
1.1 $V_I(p_\mu) = 1$ iff $I(p) \geq \mu$.
1.2 $V_I(p_\mu) = 0$ iff it is not the case that $I(p) \geq \mu$.
1.3 $V_I(\neg^k p_\mu) = V_I(\neg^{k-1} p_{\sim\mu}), k \geq 1$.
2. If A and B are formulas, then
2.1 $V_I(A \wedge B) = 1$ iff $V_I(A) = V_I(B) = 1$.
2.2 $V_I(A \vee B) = 1$ iff $V_I(A) = 1$ or $V_I(B) = 1$.
2.3 $V_I(A \to B) = 1$ iff $V_I(A) = 0$ or $V_I(B) = 1$.
3. If F is a complex formula, then
3.1 $V_I(F) = 1 - V_I(\neg_* F)$.

If $V_I(A) = 1$ for a formula A, then we say that V_I *satisfies* A; similarly if $V_I(A) = 0$, then we say that V_I *does not satisfy* A. If $V_I(A) = 1$ for any I, then we say that A is *valid*, written $\models A$.

Theorem 2.9

Let p be a propositional symbol and $\lambda, \mu, \rho \in |\tau|$. We have

1. $\models p_\perp$.
2. $\models p_\lambda \to p_\mu$, if $\lambda \geq \mu$.
3. $\models p_\lambda \wedge p_\mu \to p_\rho$, where $\rho = \lambda \vee \mu$.

Proof: 1. For any interpretation I, we have $I(p) \geq \perp$. Therefore $\models p_\perp$.

2. Let us supposed that there exists a I such that it is not the case that $\models p_\lambda \to p_\mu$, that is $\models p_\lambda$ and it is not the case that $\models p_\mu$. So, $I(p) \geq \lambda$ and not $I(p) \geq \mu$, which contradicts the hypothesis. Therefore, we have $\models p_\lambda \to p_\mu$, if $\lambda \geq \mu$.

3. Similar to the preceding, using conditions 2.1 and 2.2 of Definition 2.8.

Lemma 2.10

Let $\lambda_0 = \bigvee\{\mu \in |\tau| \mid \vdash p_\mu\}$. Then, $\vdash p_{\lambda_0}$.

Proof: The set $\{\mu \in |\tau| \mid p_\mu \in \Gamma\}$ is finite: $\{\lambda_1, \lambda_2, ..., \lambda_k\}$. Thus, $\vdash p_{\lambda_1}, \vdash p_{\lambda_2}, ..., \vdash p_{\lambda_k}$. So, $\vdash p_{\lambda_0}$.

Theorem 2.11 (Abe [1])

Let Γ be a maximal non-trivial set of formulas. Then, the characteristic function χ of Γ, that is, $\chi_\Gamma : \mathbf{F} \to \mathbf{2}$ is the valuation function of some interpretation $I : \mathbf{P} \to |\tau|$.

Proof: Let us define the function $I : \mathbf{P} \to |\tau|$ putting $I(p) = \bigvee\{\mu \in |\tau| \mid p_\mu \in \Gamma\}$. Such a function is well defined, so $p_\perp \in \Gamma$. Let $V_I : \mathbf{F} \to \mathbf{2}$ be the valuation associated to I. We have $V_I = \chi_\Gamma$. To show this, let $p_\mu \in \Gamma$. Thus $\chi_\Gamma(p_\mu) = 1$. On the other hand, it is clear that $I(p) \geq \mu$. So, $V_I(p_\mu) = 1$. If $p_\mu \notin \Gamma$, $\chi_\Gamma(p_\mu) = 0$. Also, it is not the case that $I(p) \geq \mu$, because if so, that is, $I(p) \geq \mu$, we have $p_{I(p)} \in \Gamma$ (theorem), which is a contradiction. Therefore, it is not the case that $I(p) \geq \mu$, and thus $V_I(p_\mu) = 0$.

By theorem 2.7, $\neg^k p_\mu \in \Gamma$ iff $\neg^{k-1} p_{\sim\mu} \in \Gamma$. Thus, $\chi_\Gamma(\neg^k p_\mu) = \chi_\Gamma(\neg^{k-1} p_{\sim\mu})$, where $k \geq 1$. We will show that $V_I(\neg^k p_\mu) = \chi_\Gamma(\neg^k p_\mu)$. We proceed by induction on k. If $k = 0$, it is just the previous case. Let us suppose that it is valid for $k - 1$ ($k \geq 1$). Then, $\chi_\Gamma(\neg^k p_\mu) = \chi_\Gamma(\neg^{k-1} p_{\sim\mu}) = V_I(\neg^{k-1} p_{\sim\mu}) = V_I(\neg^k p_\mu)$.

Now let A be a formula whatsoever. We proceed by induction on the number of occurrences of connectives in A. Thus, suppose that

1. A is of the form $\neg B$. Due to the previous discussion, we can suppose that B is a complex formula. So, $\chi_\Gamma(B) = V_I(B)$. If $A \in \Gamma$, then $B \notin \Gamma$, and $\chi_\Gamma(A) = 0$ and $\chi_\Gamma(B) = 1$. But, $V_I(A) = 1 - V_I(B)$. Therefore, $V_I(A) = 0$.

2. A is of the form $B \wedge C$. $A \in \Gamma$ iff $B, C \in \Gamma$. Bu induction hypothesis, $\chi_\Gamma(B) = V_I(B)$ and $\chi_\Gamma(C) = V_I(C)$. Thus, $\chi_\Gamma(A) = V_I(A)$.

The other cases are proved as in the classical cases.

Theorem 2.12

Let Γ be a set of formulas and A a formula. Then, if $\Gamma \models A$, then $\Gamma \vdash A$.

Proof: Suppose that it is not the case $\Gamma \vdash A$. Thus, $\Gamma_0 = \Gamma \cup \{\neg_* A\}$ is non-trivial. By the previous theorem, Γ_0 is contained in a non-trivial maximal set Γ. Let $V_I : \mathbf{F} \to \mathbf{2}$ be the valuation obtained from Γ. We have $V_I(A) = 1 - V_I(\neg_* A) = 0$, which is a contradiction. Thus, $\Gamma \vdash A$.

Theorem 2.13

The logic $\mathbf{P}\tau$ is decidable.

Proof: Let A be a formula. By $Sf(A)$ we denote the set of all subformulas of A. By $At(A)$ we denote the set of atomic subformulas composing A. So, by using the valuation defined above we can check in $\sharp Sf(A) - \sharp At(A)$ steps as in the classical case up to analyze $\sharp At(A)$ atomic formulas. The validity of each atomc formula is checked in $\sharp\ |\tau|$ times. So, $At(A)$ is checked in at most $k\sharp\ |\tau|\ \sharp At(A)$ times. In this way, we can check whether A is logically valid or not in a finite numer of steps. Thus, $\mathbf{P}\tau$ is shown to be decidable.

3 Annotated Modal Logics

Now we are concerned with annotated modal systems. The first annotated modal system, which is S5 type modal extension, was introduced in Abe [2]. Later, annoated modal logics were generalized by Akama and Abe [5]. Below we show the decidability of $\mathbf{S5}\tau$ and the given technique can be applied to other annotated modal systems. The language of $\mathbf{S5}\tau$ has primitive symbols of $\mathbf{P}\tau$ with the modal operator $\Box$ (necessity operator). The possibility operator $\Diamond$ can be introduced by the following definition:

Definition 3.1

Let A be a formula. Then, we put

$\Diamond A =_{def} \neg_* \Box \neg_* \Box A$.

The postulates (axiom schemata and rules of inference) of $\mathbf{S5}\tau$ are those of $\mathbf{P}\tau$, enriched with

(1) $\Box(A \to B) \to (\Box A \to \Box B)$

(2) $\Diamond A \to \Box \Diamond A$

(3) $\Box A \to A$

(4) $A / \Box A$

Here, (1) is the axiom K, (2) the axiom 5, and (3) the axiom T. (4) is the rule of inference called *necessitation*.

Theorem 3.2 (Abe [2])

In $\mathbf{S5}\tau$ all valid formulas of $\mathbf{P}\tau$ are also valid.

Theorem 3.3 (Abe [2])

$\mathbf{S5}\tau$ is non-trivial.

We next describe a Kripke semantics for $\mathbf{S5}\tau$.

Definition 3.4

A *Kripke model* K for $\mathbf{S5}\tau$ is a tuple $[W, R, I]$ where

W is a non-empty set of worlds,
R is a binary (equivalence) relation on W,
I is an interpretation function $I : W \times \mathbf{P} \rightarrow |\tau|$.

If $w \in W, p \in \mathbf{P}, \lambda \in |\tau|$, and $I(w,p) \geq \lambda$, we say that p_λ is *true* in the world w, and *false* otherwise.

Definition 3.5

If A is a formula of $\mathbf{S5}\tau$ and $w \in W$, then we define the relation $K, w \models A$ to mean K, w *forces* A by induction on A as follows:
1. If $p \in \mathbf{P}$ and $\lambda \in |\tau|$, then
1.1 $K, w \models p_\lambda$ iff $I(w,p) \geq \lambda$ (that is to say, p_λ is true in the world w)
1.2 $K, w \models \neg^k p_\lambda$ iff $K, w \models \neg^{k-1} p_{\sim\lambda}$ $(k > 0)$
2. If A and B are formulas, then
2.1 $K, w \models A \wedge B$ iff $K, w \models A$ and $K, w \models B$
2.2 $K, w \models A \vee B$ iff $K, w \models A$ or $K, w \models B$
2.3 $K, w \models A \rightarrow B$ iff it is not the case that $K, w \models A$, or $K, w \models B$
3. If F is a complex formula, then
3.1 $K, w \models \neg F$ iff it is not the case that $K, w \models F$
4. If A is a formula, then
4.1 $K, w \models \Box A$ iff $K, v \models A$ for each $v \in W$ such that $(w, v) \in R$.

Definition 3.6

Let K be a $\mathbf{S5}\tau$-model. A Kripke model K *forces* a formula A (in symbol $K \models A$), if $K, w \models A$ for each $w \in W$. A formula A is called $\mathbf{S5}\tau$*-valid*, written $\models A$, if for any $\mathbf{S5}\tau$-model $K, K \models A$.

Theorem 3.7

Let K be a Kripke model for $\mathbf{S5}\tau$. For all formulas A, B of $\mathbf{S5}\tau$ we have
1. If A is a logically valid formula of $\mathbf{P}\tau$, then $K \models A$
2. If $K \models A$ and $K \models A \rightarrow B$, then $K \models B$
3. $K \models \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
4. $K \models \Diamond A \rightarrow \Box\Diamond A$
5. $K \models \Box A \rightarrow A$
6. If $K \models A$ then $K \models \Box A$

In the proposed semantics, we can show that there are systems $\mathbf{S5}\tau$ such that we have "inconsistent" worlds, "paracomplete" worlds, or both. Thus, $\mathbf{S5}\tau$ can be regarded as paraconsistent, paracomplete and non-alethic systems.

Theorem 3.8

Let U be a maximal non-trivial (with respect to inclusion of sets) subsets of the set of formulas$F$. Let A and B be formulas whatsoever. Then
1. If A is an axiom of $\mathbf{S5}\tau$, then $A \in U$.
2. $A \wedge B \in U$ iff $A \in U$ and $B \in U$.
3. $A \vee B \in U$ iff $A \in U$ or $B \in U$.
4. $A \rightarrow B \in U$ iff $A \notin U$ or $B \in U$.
5. If $p_\lambda, p_\mu \in U$, then $p_\theta \in U$, where $\theta = max(\lambda, \mu)$.
6. $\neg^k p_\mu \in U$ iff $\neg^{k-1} p_{\sim\mu} \in U$.
7. If $A, A \rightarrow B \in U$, then $B \in U$.
8. $A \in U$ iff $\neg_* A \notin U$. Moreover, $A \in U$ or $\neg_* A \in U$.

Proof: Let us show only 5. In fact, if $p_\lambda, p_\mu \in U$, then $p_\lambda \wedge p_\mu \in U$, by 2. But it is an axiom of the form $p_\lambda \wedge p_\mu \rightarrow p_\theta$, where $\theta = max(\lambda, \mu)$. It follows that $p_\lambda \wedge p_\mu \rightarrow p_\theta \in U$, and so $p_\theta \in U$, by 7.

Given a set U of formulas, define$U_\Box = \{A \mid \Box A \in U\}$. Let us consider the canonical model $K^c = [W^c, R^c, V^c]$, where $W^c = \{U \mid U$ is a maximal non-trivial set$\}$, $I^c : W^c \times \mathbf{P} \rightarrow |\tau|$, defined by $I^c(U, p) =_{def} \{\mu \in |\tau| \mid p_\mu \in U\}$. Such a function is well defined, so $p_\perp \in U$. Moreover, we define $R^c =_{def} \{(U, U') \mid U_\Box \subseteq U'\}$.

Lemma 3.9

For all propositional variable p, if U is a maximal non-trivial set of formulas, then we have $p_{I(U,p)} \in U$.

Proof: It is a simple consequence of the item 5 of the previous theorem.

Theorem 3.10

For any formula A and for any non-trivial maximal set U, we have $(K, U) \models A$ iff $A \in U$.

Proof: Let us suppose that A is of the form p_λ and $(K, U) \models p_\lambda$. It is clear by the previous lemma that $p_{I(U,p)} \in U$. It also follows that $I(U, p) \geq \lambda$. It is an axiom of the form $p_{I(U,p)} \rightarrow p_\lambda$. Thus, $p_\lambda \in U$. Now, let us suppose that $p_\lambda \in U$. By the previous lemma, $p_{I(U,p)} \in U$. It follows that $I(U, p) \geq \lambda$. Thus, by definition, $(K, U) \models p_\lambda$. By theorem 3.8, $\neg^k p_\lambda \in U$ iff $\neg^{k-1} p_{\sim\lambda} \in U$. Thus, by definition 3.5, $(K, U) \models \neg^k p_\lambda$ iff $(K, U) \models \neg^{k-1} p_{\sim\lambda}$. So, by induction on k, the assertion is true for hyper-literals. For the other cases, the proof is similar to that of the classical case.

Corollary 3.10.1

For any formula A, $\models A$ iff $\vdash A$.

In what follows, if $K = [W, R, I]$ is a Kripke model for $\mathbf{S5}\tau$, $|K|$ indicates the number of worlds $\sharp W$ added with the number $\sharp R$.

Theorem 3.11

There is an algorithm that, given a finite structure K, a world $w \in W$, and a formula A, can determine whether $K, w \models A$, in time $k\ |k|\ \sharp|\tau|\ \sharp Sf(A)$.

Proof: Let $A_1, A_2, ..., A_m$ be the subformulas of A and $A_m = A$ if A_i is a subformula of A_j, where $i < j$. We can label each world in W with A_j or $\neg_* A_i$, for $i = 1, 2, ..., k$, by induction on k, according if A_j is true or not in w, in time $ck\ |K|$ for some constant c. If A_{k+1} is the form $\Box A_j$, where $j < k+1$, we lable a world w with $\Box A_j$ iff each world w' such that $(w, w') \in R$ is labeled with A_j or $\neg_* A_j$. This step can be clearly carried out in time $m\ |K|\ \sharp|\ \tau\ |$.

Definition 3.12

A formula A is said to be a *non-trivial formula* if it does not satisfy the condition that $A, \neg A \vdash B$ for arbitrary formula B.

Theorem 3.13

If A is a non-trivial formula, then A is satisfiable in a structure K with at most $2^{2|A|}$ worlds.

Proof: Let $Sf(A)^* = Sf(A) \cup \{\neg A \vdash A \in Sf(A)\}$. Let $Nt(A)$ be the set of maximal non-trivial subsets of $Sf(A)^*$. Each subset of $Sf(A)^*$ can be extended to an element of $Nt(A)$. Moreover, a member of $Nt(A)$ contains at most $2\ |A|$ elements. So the cardinality of $Nt(A)$ is at most $2^{2|A|}$. We can thus construct a structure $K_A = [W_A, R, I]$ similar to the theorem 3.10 except that we take $w_A = \{w_v \mid v \in Nt(A)\}$. We can then show that if $V \in Nt(A)$, then for all $B \in Sf(A)^*$ we have $(K_A, w_v) \models B$ iff $B \in V$.

From theorems 3.11 and 3.13, the decidability of $\mathbf{S5}\tau$ follows:

Theorem 3.14

The logic $\mathbf{S5}\tau$ is decidable.

4 Concluding Remarks

We have semantically established the decidability of the propositional annotated logic $\mathbf{P}\tau$ and its S5 type modal extension $\mathbf{S5}\tau$. The decidability of other modal systems can be obtained in a similar way. However, if τ is an infinite lattice, resulting annotated systems are generally undecidable. These results seem important since annotated systems are designed to deal with many issues in computer science.

However, there are several open problems. First, it is necessary to work out the so-called *filtration* to prove the decidability of modal systems. Because filtration is studied for most classical modal systems, it would be possible to modify the existing methods for annotated modal logics. Second, it is of interest to prove the decidability theorem in a proof-theoretic or algebraic manner. For this purpose, we need to formulate a sequent calculus for annotated systems. An algebraic proof of the decidability of annotated systems could be developed by using the technique in Abe [3]. Third, we should investigate whether different versions of annotated logics are decidable or not. For instance, *generalized annotated logics* of Kifer and Subrahmanian [10] are based on rich algebraic structures on τ. *General annotated logics* of Sylvan and Abe [12] allow an annotation at all levels; also see [7]. Akama and Abe [6] proposed *fuzzy annotated logics* to model fuzzy reasoning. In addition, we must search the decidable classes of annotated predicate logics for applications; see Abe [1, 4]. We hope to report these issues in forthcoming papers.

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