Formula-Preferential Systems for Paraconsistent Non-Monotonic Reasoning (an extended abstract)

Arnon Avron and Iddo Lev
School of Computer Science
Tel-Aviv University
Ramat Aviv 69978, Israel
email: {aa,iddolev}@post.tau.ac.il

Abstract

We provide a general framework for constructing natural consequence relations for paraconsistent and plausible nonmonotonic reasoning. The framework is based on preferential systems whose preferences are based on the satisfaction of formulas in models. The framework encompasses different types of preferential systems that were developed from different motivations of paraconsistent reasoning and non-monotonic reasoning, and reveals an important link between them.

1 Introduction

For a long time the research efforts on paraconsistency and on nonmonotonic reasoning were separated. The former research dealt with the question of how to prevent the inference of every fact from an inconsistent source of knowledge, and how to isolate inconsistent parts of the knowledge and yet work in the usual way with the consistent parts. The latter dealt with the question of how to "jump to conclusions" based on partial knowledge of the domain (this is needed since having complete knowledge is often unrealistic), and how to revise previous "hasty" conclusions in the face of new and fuller information.

However, in recent years the formal connections between these two areas have begun to be revealed. It is only natural that such a connection would exist, because conclusions that are drawn based on partial knowledge may contradict new and more reliable information, and each new piece of information may contradict previous information and hence force us to revise some of our knowledge. As the famous example goes, if we conclude that Tweety can fly based on the sole fact that it is a bird, the new piece of information that Tweety is a penguin and penguins cannot fly forces us not only to revise previous conclusions but also to deal with the fact that we now have a contradiction in our knowledge.

Both goals of handling contradictions and reasoning nonmonotonically require some selection between alternatives: which parts of the knowledge to retain and which to discard or change. A central tool in both fields has been *preferential systems*, meaning that only a subset of the models should be relevant for making inferences from a theory. These models are the most preferred ones according to some criterion.

In the research on paraconsistency, preferential systems were used for constructing logics which are paraconsistent but stronger than substructural paraconsistent logics. The preferences in these systems were defined in different ways. Some were based on checking which abnormal formulas (such as $\psi \wedge \neg\psi$) are satisfied in the models of a theory (see e.g. [Priest, 1991; Batens, 1998]). Others were based on preferences between the truth values that are assigned to formulas (see e.g. [Kifer and Lozinskii, 1992; Arieli and Avron, 2000a]).

Preferential systems were also used for providing semantics for nonmonotonic consequence relations (see e.g. [Shoham, 1987; Kraus *et al.*, 1990; Makinson, 1994]). It was discovered, however, that in order for them to satisfy all the desired theoretical properties that plausible nonmonotonic relations should have (see e.g. [Lehmann, 1992]), preferential systems need to satisfy a further condition called stopperedness or smoothness. The problem is that this condition is usually not easy to verify.

In this paper we provide a general framework for constructing natural consequence relations for paraconsistent and plausible nonmonotonic reasoning. The main technique is using preferential systems in which the preference between models is made according to a certain set of formulas which are satisfied in them. The framework encompasses different types of preferential systems that were used for constructing useful paraconsistent consequence relations. Moreover, these natural preferential systems that were originally designed for paraconsistent reasoning satisfy the stopperedness condition as well, and hence have also the desired theoretical properties of nonmonotonic consequence relations.

As we said, the theoretical research on nonmonotonic reasoning and the research on paraconsistent reasoning have been conducted separately at first. Nevertheless, formula-preferential systems, which are a generalization of methods used in the latter, solve a key issue in the former, and help to bridge the gap between the two directions of research and to combine them under a unified framework. This provides strong evidence for their important rule in non-classical reasoning.

2 Preliminaries

In what follows $\mathcal{L}$ is a language, $\mathcal{W}$ is its set of wffs, ψ, ϕ, τ denote formulas of $\mathcal{L}$, and Γ, Δ denote sets of formulas.

When the language is propositional, $\mathcal{A}$ denotes its set of propositional variables, and p, q, r denote such variables.

Definition 2.1 [1] A *semantic structure* for a language $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathcal{M}_\mathcal{S}, \models^\mathcal{S} \rangle$, where $\models^\mathcal{S} \subseteq \mathcal{M}_\mathcal{S} \times \mathcal{W}$. $\mathcal{M}_\mathcal{S}$ is a set of *models* and $\models^\mathcal{S}$ is called a *satisfaction relation*. A model $m \in \mathcal{M}_\mathcal{S}$ *satisfies* a formula ψ if $m \models^\mathcal{S} \psi$. m is a *model of* Γ ($m \models^\mathcal{S} \Gamma$) if it satisfies every formula in Γ. The set of the models of Γ is denoted by $mod(\Gamma, \mathcal{S})$. Δ is a *consequence* of Γ in $\mathcal{S}$ ($\Gamma \vdash^\mathcal{S} \Delta$) if for every $m \in mod(\Gamma, \mathcal{S})$, $m \models^\mathcal{S} \phi$ for some $\phi \in \Delta$.

It is easy to verify that every semantic structure induces a monotonic consequence relations (*mcr* in short, i.e. it satisfies reflexivity: if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$, monotonicity: if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ then $\Gamma' \vdash \Delta'$, and cut: if $\Gamma \vdash \psi, \Delta$ and $\Gamma', \psi \vdash \Delta'$ then $\Gamma, \Gamma' \vdash \Delta, \Delta'$).

A common type of semantic structures for propositional logics is the class of multi-valued matrices. In these structures the value that a valuation assigns to a complex formula is uniquely determined by the values that it assigns to its subformulas. However, an agent acting in the real world often has only incomplete or inconsistent knowledge to guide its decisions. One possible approach for dealing with this problem is to borrow the idea of *non-deterministic* computations from automata and computability theory, and apply it for assigning truth-values to complex formulas. Here we use a natural generalization of the logical concept of a matrix – the value that a valuation assigns to a complex formula can be chosen non-deterministically from a certain nonempty set of options:

Definition 2.2 [Avron and Lev, 2000] A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{S} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{T}$ is a non-empty set of *truth values*, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{T}$ (its *designated values*), and for every n-ary connective $\diamond$, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{T}^n$ to $2^\mathcal{T} - \{\emptyset\}$. A *valuation* in $\mathcal{S}$ is a function $v : \mathcal{W} \to \mathcal{T}$ that satisfies the condition: if $\diamond$ is an n-ary connective, and $\psi_1, \ldots, \psi_n \in \mathcal{W}$, then $v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$. $\mathcal{V}_\mathcal{S}$ denotes the set of valuations of $\mathcal{S}$. The satisfaction relation $\models^\mathcal{S} \subseteq \mathcal{V}_\mathcal{S} \times \mathcal{W}$ is defined: $v \models^\mathcal{S} \psi$ iff $v(\psi) \in \mathcal{D}$. We identify the Nmatrix $\mathcal{S}$ with the semantic structure $\langle \mathcal{V}_\mathcal{S}, \models^\mathcal{S} \rangle$. Every (deterministic) matrix can be identified with an Nmatrix whose functions in $\mathcal{O}$ always return singletons.

In addition to their obvious potential for reasoning under uncertainty and for specification and verification of non-deterministic programs, N-matrices have considerable practical technical applications. It is well known that every propositional logic can be characterized semantically using a multi-valued matrix ([Łos and Suszko, 1958]). However, there are important logics whose characteristic matrices necessarily consist of an infinite number of truth values, and are thus of little help in providing decision procedures for their logics, or in getting real insight into them. Our generalization of the concept of a matrix allows us to replace in many cases an infinite characteristic matrix for a given propositional logic by a characteristic *finite* structure that automatically provides a decision procedure.

[1] See e.g. [Makinson, 1994; Lehmann, 1992].

A prime example for such a case is the Nmatrix $\mathcal{S}_p^\top$. It is defined in the classical propositional language with the connectives $\{\wedge, \vee, \supset, \neg, f\}$. The interpretation of all the connectives except for $\neg$ is the classical one (e.g. $x \widetilde{\wedge} y = \{t\}$ if $x = y = t$ and $\{f\}$ otherwise), whereas negation is a non-deterministic operation: $\widetilde{\neg} f = \{t\}$ but $\widetilde{\neg} t = \{t, f\}$ (a formula and its negation may both be assigned t in a valuation). The mcr induced by $\mathcal{S}_p^\top$ can be characterized using the Gentzen-type calculus that is obtained from Gentzen's original calculus (in [Gentzen, 1969]) for classical logic (including cut) by omitting the rule $[\neg \Rightarrow]$ for introducing negation on the left. The consequence relation induced by $\mathcal{S}_p^\top$ cannot be induced by any finite matrix. Moreover, any two-valued Nmatrix which has at least one proper nondeterministic operation does not have an equivalent finite matrix.

We shall later mention nonmonotonic logics which are based on the underlying paraconsistent monotonic logics induced by $\mathcal{S}_p^\top$ as well as the well-known four-valued matrix $\mathcal{S}_4$ of [Belnap, 1977] with the truth-values $\{t, f, \top, \bot\}$ (in the classical propositional language) and its submatrix $\mathcal{S}_3^\top$ with the values $\{t, f, \top\}$ ($t, \top$ are designated). $\mathcal{S}_3^\top$ is the *maximal* paraconsistent logic that contains $\vdash_{\rm pos}$ (positive classical logic), whereas $\mathcal{S}_p^\top$ is the *minimal* logic that contains $\vdash_{\rm pos}$ and in which $\vdash_p^\top \psi, \neg\psi$ for all ψ. [2]

The following result will be important for our framework in section 4:

Theorem 2.3 [3] *Every finite Nmatrix is finitary.*

3 Nonmonotonic Consequence Relations

In recent years there has been a wide study of theoretical properties that nonmonotonic consequence relations should satisfy (see e.g. [Arieli and Avron, 2000b] for a list of such works). Here we shall use the following notion:

Definition 3.1 [4] Let $\vdash$ be an mcr. A binary relation $\mid\!\sim$ between sets of formulas and sets of formulas is called $\vdash$-*plausible* if it satisfies the following conditions:

Ext $\vdash$-*extension*:
for every $\Gamma, \Delta \neq \emptyset$, if $\Gamma \vdash \Delta$ then $\Gamma \mid\!\sim \Delta$.
RM *right monotonicity*:
if $\Gamma \mid\!\sim \Delta$ and $\Delta \subseteq \Delta'$ then $\Gamma \mid\!\sim \Delta'$.
LCM *left cautious monotonicity*:
if $\Gamma \mid\!\sim \psi$ for every $\psi \in \Gamma'$, and $\Gamma \mid\!\sim \Delta$ then $\Gamma, \Gamma' \mid\!\sim \Delta$.
LCC *left cautious cut*:
if $\Gamma \mid\!\sim \psi, \Delta$ for every $\psi \in \Sigma$ and $\Gamma, \Sigma \mid\!\sim \Delta$ then $\Gamma \mid\!\sim \Delta$.
RCC *right cautious cut*:
if $\Gamma, \psi \mid\!\sim \Delta$ for every $\psi \in \Sigma$ and $\Gamma \mid\!\sim \Sigma, \Delta$ then $\Gamma \mid\!\sim \Delta$.

A central method for providing semantics to plausible nonmonotonic consequence relations has been the use of preferential systems. The idea of preferential systems (which began in [Shoham, 1987]) is that instead of using all the models of a given theory for checking which conclusions follow from

[2] The mcr induced by $\mathcal{S}_p^\top$ is the same as **CLuN** from [Batens *et al.*, 1999] and the logic $K/2$ of [Béziau, 1999].

[3] See [Avron and Lev, 2000].

[4] After [Lehmann, 1992; Arieli and Avron, 2000b], with slightly different names and conditions.

it, the models are ordered by a preference relation, and only the most preferred models are used as relevant for making inferences from the theory.

Notation 3.2 If A is a set with a pre-order $\preceq$, $x \prec y$ denotes $x \preceq y$ and $y \not\preceq x$. $\text{Min}_{\preceq}(A) = \{x \in A \mid \forall y \in A.\, y \not\prec x\}$.

Definition 3.3 [5] Let $\mathcal{S}$ be a semantic structure.

1. A *preferential system* in $\mathcal{S}$ is a pair $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$, where $\preceq$ is a pre-order on $\mathcal{M}_{\mathcal{S}}$.
2. A model $m \in mod(\Gamma, \mathcal{S})$ is a $\mathcal{P}$*-preferential model* of Γ if $m \in pmod(\Gamma, \mathcal{P}) = \text{Min}_{\preceq}(mod(\Gamma, \mathcal{S}))$.
3. A set of formulas Γ $\mathcal{P}$*-preferentially entails* a set of formulas Δ (notation: $\Gamma \vdash^{\mathcal{P}} \Delta$) if for every $m \in pmod(\Gamma, \mathcal{P})$ there is a $\phi \in \Delta$ s.t. $m \models^{\mathcal{S}} \phi$.[6] $\vdash^{\mathcal{P}}$ is called the *consequence relation*[7] *induced by* $\mathcal{P}$.

Definition 3.4 Let A be a set with a pre-order $\preceq$. A is *stoppered under* $\preceq$ if every $x \in A$ has $x' \in \text{Min}_{\preceq}(A)$ s.t. $x' \preceq x$.

Definition 3.5 [8] A preferential system $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ is *stoppered* if for all Γ, $mod(\Gamma, \mathcal{S})$ is stoppered under $\preceq$.

Theorem 3.6 [9] *If $\mathcal{P}$ is a stoppered preferential system in $\mathcal{S}$ then $\vdash^{\mathcal{P}}$ is $\vdash^{\mathcal{S}}$-plausible.*

Note: The stopperedness condition is introduced because some preferential systems which are not stoppered do not satisfy the condition LCM of Definition 3.1 (although the other conditions are always satisfied by all preferential systems).

As noted in [Kraus *et al.*, 1990; Makinson, 1994], it is usually not easy to check whether a preferential system is stoppered. Preferential systems were originally developed as a framework for providing semantics for nonmonotonic inference relations. They were also used, apparently independently at first, for constructing systems for reasoning with inconsistencies (and other abnormalities) in a way which is on the one hand non-trivial and on the other hand not as weak as monotonic substructural logics (see e.g. [Priest, 1991; Kifer and Lozinskii, 1992; Batens, 1998]). Interestingly, these ideas, which were developed from motivations different from stopperedness will provide us with methods for constructing stoppered preferential systems.

4 Formula-Preferential Systems

Formula-preferential systems are a generalization of the 'minimal-abnormality strategy" from [Batens, 1998]. That paper uses a specific selection of models from $\mathcal{S}_p^{\top}$. Denoting $K(v) = \{\psi \in \mathcal{W}_{\text{cl}} \mid v(\psi \wedge \neg\psi) = t\}$, a model v of Γ is selected iff there is no other model v' of Γ s.t. $K(v') \subset K(v)$. In this way the minimal-abnormality strategy minimizes the abnormalities (here – inconsistencies) in the models of a theory (by "abnormality" we mean a formula that leads to triviality w.r.t. a desired logic, here – classical logic).

Our generalization is to choose some set G of formulas in the language, and to have the preferential system select those models of a theory that minimize the satisfaction of formulas from G. Formula-preferential systems can be defined w.r.t. any set G of formulas, and also in any semantic structure, since what is important for the preference relation between the models is which formulas from G they satisfy, and not their inner structure. Formally:

Notation 4.1 Let $\mathcal{S}$ be a semantic structure and let $G \subseteq \mathcal{W}$. For $m \in \mathcal{M}_{\mathcal{S}}$ denote: $\mathsf{Sat}_{\mathcal{S},G}(m) = \{\psi \in G \mid m \models^{\mathcal{S}} \psi\}$.

Definition 4.2 Let $G \subseteq \mathcal{W}$. A *formula-preferential system based on* G is a preferential system $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ that satisfies: for all $m_1, m_2 \in \mathcal{M}_{\mathcal{S}}$, $m_1 \preceq m_2$ iff $\mathsf{Sat}_{\mathcal{S},G}(m_1) \subseteq \mathsf{Sat}_{\mathcal{S},G}(m_2)$. $\mathcal{P}$ is called in short a "G-preferential system".

In this way formula-preferential systems provide a natural source of stoppered preferential systems. The formulas in G express the undesired properties which we would like to minimize in the preferred models, and the nonmonotonic consequence relations that these systems induce satisfy the conditions of Definition 3.1 whenever they are based on a finitary semantic structure:

Theorem 4.3 *If $\mathcal{P}$ is a formula-preferential system in a finitary semantic structure then $\mathcal{P}$ is stoppered.*

Corollary 4.4 *If $\mathcal{P}$ is a formula-preferential system in a finitary semantic structure $\mathcal{S}$ then $\vdash^{\mathcal{P}}$ is $\vdash^{\mathcal{S}}$-plausible.*

Corollary 4.5 *If $\mathcal{P}$ is a formula-preferential system in a finite Nmatrix $\mathcal{S}$ then $\vdash^{\mathcal{P}}$ is $\vdash^{\mathcal{S}}$-plausible.*

The last Corollary follows from Theorem 2.3 and Corollary 4.4. Since in practice one usually works with finite structures, this means that this result has great practical significance.

We mention now some known systems from the literature which can be constructed using formula-preferential systems. All of them are based on finite Nmatrices, so by Corollary 4.5 their induced consequence relations are plausible. Whenever the underlying monotonic logics are paraconsistent, so are the induced nonmonotonic relations.

Closed-World Assumption

In the "Closed-World Assumption" method [Reiter, 1978], a propositional variable that cannot be proved is assumed to be false. A corresponding formula-preferential system is defined in the classical two-valued matrix and is based on $\mathcal{A}_{\text{cl}}$ (the propositional variables of the language). The obtained consequence relation is nonmonotonic but not paraconsistent.

Preferential systems for handling contradictions

$\vdash_3^{\top}$ is paraconsistent, but it is too weak for adequate reasoning, e.g. the Disjunctive Syllogism (from ψ, $\neg\psi \vee \phi$ infer ϕ) is not valid in it, even on classically consistent sets. A consequence relation that is located between $\vdash_3^{\top}$ and classical logic can be obtained by using the formula-preferential system $\mathcal{P} = \langle \mathcal{S}_3^{\top}, \preceq \rangle$ that is based on $G = \{p \wedge \neg p \mid p \in \mathcal{A}_{\text{cl}}\}$. $\vdash^{\mathcal{P}}$ is the same as **LPm** of [Priest, 1991] (when $\mathcal{S}_3^{\top}$ is without $\supset$) and **ACLuNs2** of [Batens, 1998]. It is nonmonotonic,

[5] Following [Makinson, 1994; Lehmann, 1992].

[6] Note that we do *not* require that $m \in pmod(\{\phi\}, \mathcal{P})$, or that $m \in pmod(\Gamma \cup \{\phi\}, \mathcal{P})$.

[7] The term "consequence relation" here is more general than an mcr. In particular, we do not assume monotonicity.

[8] Following [Makinson, 1994]. In [Kraus *et al.*, 1990; Lehmann, 1992] this is called *smoothness*.

[9] A Generalization of a result in [Arieli and Avron, 2000b].

paraconsistent, and in contrast to $\vdash_3^{\top}$, it is the same as classical logic on classically consistent sets.

Adaptive Logics

Adaptive logics [Batens, 1998; 2000] were originally introduces by dynamic proof systems that are designed to mimic some aspects of human reasoning with inconsistencies, especially the fact that conclusions that are drawn at a certain stage may be rejected at a later stage because of other conclusions, and then even accepted again. The name "adaptive" is due to the fact that these logics adapt their rules to the given set of premises. E.g. the Disjunctive Syllogism is not valid in $\vdash_3^{\top}$. Its use is not allowed by **ACLuNs2** on $\Gamma = \{r, \neg r, \neg r \vee s, p, \neg p \vee q\}$ for inferring s (since r behaves inconsistently) but it is allowed for inferring q (since there is no reason to suppose that p behaves inconsistently).

Adaptive logics that are based on the minimal-abnormality strategy are a special case of the formula-preferential systems where the set G is taken as a set of abnormal formulas. For example, **ACLuN2** (note: not **ACLuNs2**) is induced by the formula-preferential system in $\mathcal{S}_p^{\top}$ that is based on $G = \{\psi \wedge \neg\psi \mid \psi \in \mathcal{W}_{\rm cl}\}$. **ACL∅2** from [Batens, 1999] is based on the two-valued Nmatrix $\mathcal{S}_0$ in which all the connectives of the classical language are weakened: for an n-ary connective $\diamond \in \Sigma_{\rm cl} = \{\wedge, \vee, \supset, \neg, f\}$ and any $\bar{x} \in \{t, f\}^n$, $\widetilde{\diamond}(\bar{x}) = \{t, f\}$. $\mathcal{S}_0$ has in addition the connectives $\sim$ and & which function in $\mathcal{S}_0$ as classical negation and conjunction. $\mathcal{P}_0$ is the formula-preferential system in $\mathcal{S}_0$ that is based on the set G_0: the set that includes all formulas which express the fact that a certain formula $\diamond(\psi_1, \ldots, \psi_n)$ and one or more of $\psi_1, \ldots, \psi_n$ are assigned values that are illegal in a classical valuation, e.g. $\psi \,\&\, \neg\psi$, $(\psi \,\&\, \phi) \,\&\, \sim(\psi \wedge \phi)$, $(\psi \,\&\, \sim\phi) \,\&\, (\psi \supset \phi)$, etc. In comparison to **ACLuN2**, **ACL∅2** is "adaptive" on all the connectives in $\Sigma_{\rm cl}$, not only $\neg$.

Other adaptive logics (see [Batens, 2000]) use a formula-preferential system $\mathcal{P}$ in a more complicated way: the definition of the adaptive logic $\mid\!\sim$ is: $\Gamma \mid\!\sim \Delta$ iff $Tr(\Gamma) \vdash^{\mathcal{P}} Tr(\Delta)$, where Tr is some pre-processing of the formulas.

5 Pointwise-preferential systems

[Arieli and Avron, 2000b] suggests another method for constructing preferential systems that are stoppered. The method is based on a type of preferential systems called *pointwise* preferential systems. The underlying idea is to have a preference between the truth values of a multiple-valued structure and to base the preference between the valuations on this preference. For example, in the (N)matrix $\mathcal{S}_4$, we might prefer the classical values t and f over $\top$ and $\bot$, since a valuation satisfies exactly one of ψ and $\neg\psi$ iff it assigns a classical value to ψ. If there are two models for a given set of premises and they assign the same values to all atomic formulas except that one assigns t to p and the other $\top$, we might prefer the first. This is the underlying idea of the following definition:

Definition 5.1 [10] Let $\mathcal{S}$ be an Nmatrix with a set of truth values $\mathcal{T}$, and let $\leq$ be a pre-order on $\mathcal{T}$. A *pointwise preferential system* (in $\mathcal{S}$) *based on* $\leq$ is a preferential system $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ that satisfies the condition: for all $v_1, v_2 \in \mathcal{V}_{\mathcal{S}}$, $v_1 \preceq v_2$ iff for every propositional variable p, $v_1(p) \leq v_2(p)$. If $\leq$ is a partial-order, $\mathcal{P}$ is called *strongly pointwise*. $\mathcal{P}$ will be called in short a "$\leq$-preferential system".

[10]A generalization of 'pointwise preferential systems' from [Arieli and Avron, 2000b], which are in our notations strongly pointwise preferential systems in matrices.

[Arieli and Avron, 2000b] shows that pointwise preferential systems that are based on well-founded partial orders are stoppered and hence induce plausible relations.

Pointwise preferential systems are in general a different type of systems than formula-preferential systems. Nevertheless, by adding certain connectives to the language, we can construct for each pointwise preferential system a formula-preferential system that induces the same consequence relation and, in a certain sense, has the same preference relation. A consequence of this embedding is that the finitariness of the underlying semantic structure ensures the stopperedness property:

Definition 5.2 Let $\mathcal{S} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be a Nmatrix for a propositional language $\mathcal{L}$, and let $\mathcal{L}'$ be a propositional language with the same variables as $\mathcal{L}$ but with additional logical connectives. An *extension of* $\mathcal{S}$ *to* $\mathcal{L}'$ is a Nmatrix $\mathcal{S}' = \langle \mathcal{T}, \mathcal{D}, \mathcal{O}' \rangle$ for $\mathcal{L}'$ s.t. $\mathcal{O}' \supseteq \mathcal{O}$. A valuation v' in $\mathcal{S}'$ is an *extension* of a valuation v in $\mathcal{S}$ to $\mathcal{L}'$ if v and v' agree on $\mathcal{W}$.

Definition 5.3 Let $\mathcal{S} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$ and let $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ be a $\leq$-preferential system. A formula-preferential system *associated* with $\mathcal{P}$ is $\mathcal{P}' = \langle \mathcal{S}', \preceq' \rangle$ for the language $\mathcal{L}'$, where $\mathcal{L}'$ is like $\mathcal{L}$ but with the added or defined connectives $\{I_x \mid x \in \mathcal{T}\}$, $\mathcal{S}'$ is an extension of $\mathcal{S}$ to $\mathcal{L}'$ with the same truth values s.t. for every $x, y \in \mathcal{T}$, $\widetilde{I_x} y \subseteq \mathcal{D}$ if $y \geq x$ and $\widetilde{I_x} y \subseteq \mathcal{T} - \mathcal{D}$ otherwise, and $\mathcal{P}'$ is based on $G = \{I_x p \mid x \in \mathcal{T},\ p \in \mathcal{A}\}$.

Note: For all valuations v in $\mathcal{S}'$, $v \models^{\mathcal{S}'} I_x\psi$ iff $v(\psi) \geq x$.

Theorem 5.4 *Let $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ be a $\leq$-preferential system and let $\mathcal{P}' = \langle \mathcal{S}', \preceq' \rangle$ be an associated formula-preferential system.*

1. *For all $\Gamma, \Delta \subseteq \mathcal{W}$, $\Gamma \vdash^{\mathcal{P}} \Delta$ iff $\Gamma \vdash^{\mathcal{P}'} \Delta$.*
2. *For all $v_1, v_2 \in \mathcal{V}_{\mathcal{S}}$, $v_1 \preceq v_2$ iff for each of their (respective) extensions $v_1', v_2' \in \mathcal{V}_{\mathcal{S}'}$ to $\mathcal{L}'$, $v_1' \preceq' v_2'$.*

Note: for each $x \in \mathcal{T}$ that is a least element ($x \leq y$ for all $y \in \mathcal{T}$), defining G without any formula $I_x p$ will give the same result, since such x guarantees that $v \models^{\mathcal{S}'} I_x p$ for all v, and so the presence of these formulas in G does not influence the preference relation.

Corollary 5.5 *If $\mathcal{P}$ is a pointwise preferential system in a finitary Nmatrix $\mathcal{S}$ then $\mathcal{P}$ is stoppered and $\vdash^{\mathcal{P}}$ is $\vdash^{\mathcal{S}}$-plausible.*

The pointwise preferential systems from [Arieli and Avron, 2000a] are based on the finite matrices $\mathcal{S}_4$ and $\mathcal{S}_3^{\top}$, so they can be embedded in formula-preferential systems, and their induced consequence relations are therefore plausible.

The first type of systems is based on the idea of minimal knowledge. In $\mathcal{S}_4$, it is based on the partial order $\leq_k$: $\bot <_k (t, f) <_k \top$. The corresponding formula-preferential system (in $\mathcal{S}_4$) is based on the set that includes $I_{\top} p = p \wedge \neg p$, $I_t p =$

p, $I_f p = \neg p$, $I_\perp p = \phi \supset \phi$, for all $p \in \mathcal{A}_{\rm cl}$. According to the remark above, $I_\perp p$ is redundant. In this particular case $I_\top p$ is also redundant, i.e. it is enough to take $G = \mathcal{A}_{\rm cl} \cup \{\neg p \mid p \in \mathcal{A}_{\rm cl}\}$. Note that contrary to the original motivation behind the minimal-abnormality strategy of [Batens, 1998], in this system (as well as in CWA of section 4) we do not regard the formulas in G as abnormal (in particular, all the variables are in G), but rather as the formulas whose satisfaction we want to minimize in the models.

The second type of systems from [Arieli and Avron, 2000a] is based on the idea of minimal inconsistency: it is based on choosing a subset of inconsistent truth values $\mathcal{I}$, and defining the pre-order $\leq_{\mathcal{I}}$: $x_1 \leq_{\mathcal{I}} x_2$ iff $x_1 \in \mathcal{T} - \mathcal{I}$ or $x_2 \in \mathcal{I}$. The preferential system from section 4 that corresponds to **ACLuNs2** can be defined as the $\leq_{\mathcal{I}}$-preferential system in $\mathcal{S}_3^\top$ where $\mathcal{I} = \{\top\}$. If $\mathcal{T} = \{t, f, \top\}$, this is the only inconsistency set. In the general case, there may be other inconsistency sets. For example, in $\mathcal{S}_4$, both $\mathcal{I}_1 = \{\top\}$ and $\mathcal{I}_2 = \{\top, \perp\}$ are inconsistency sets, and they induce different consequence relations. Let $\mathcal{P}_i$ $(i = 1, 2)$ be the pointwise $\leq_{\mathcal{I}_i}$-preferential system in $\mathcal{S}_4$. $\mathcal{P}_1$ can be embedded in the formula-preferential system (in $\mathcal{S}_4$) that is based on the set G that includes $I_\top p = p \wedge \neg p$ for all $p \in \mathcal{A}_{\rm cl}$ and $I_t p = I_f p = I_\perp p = \phi \supset \phi$. According to the remark above, only the formulas $I_\top p$ are necessary. For $\mathcal{P}_2$, the set includes $I_\top p = I_\perp p = (p \supset \neg p) \wedge (\neg p \supset p)$ for all $p \in \mathcal{A}_{\rm cl}$ (and $I_t p = I_f p = \phi \supset \phi$ are redundant).

6 Conclusion

Our main goal in this paper was to demonstrate the central role of formula-preferential systems in non-classical reasoning. We have shown how different systems from the literature for reasoning in the face of inconsistencies and other abnormalities, can be constructed in this framework. Moreover, although most of these systems were not originally part of the theoretical research of nonmonotonic consequence relations, the generalization of their preference relations to the idea of formula-preferential systems provides us with a method for ensuring the condition of stopperedness: formula-preferential systems that are based on finitary semantic structures are stoppered, and hence satisfy theoretical desiderata for a plausible nonmonotonic logic. All the examples from the literature that we have given are of this kind since they are based on finite non-deterministic matrices.

References

[Arieli and Avron, 2000a] O. Arieli and A. Avron. Bilattices and paraconsistency. In [Batens *et al.*, 2000], pages 11–28. 2000.

[Arieli and Avron, 2000b] O. Arieli and A. Avron. General patterns for nonmonotonic reasoning: from basic entailments to plausible relations. *Logic Journal of the IGPL*, 8(2):119–148, 2000.

[Avron and Lev, 2000] A. Avron and I. Lev. Non-deterministic matrices. 2000. Submitted.

[Batens *et al.*, 1999] D. Batens, K. De Clercq, and N. Kurtonina. Embedding and interpolation for some paralogics. The propositional case. (to appear in *Reports on Mathematical Logic*), 1999.

[Batens *et al.*, 2000] D. Batens, C. Mortensen, G. Priest, and Jan Paul Van Bendegem, editors. *Frontiers of Paraconsistent Logic*. King's College Publications, Research Studies Press, Baldock, UK, 2000.

[Batens, 1998] D. Batens. Inconsistency-adaptive logics. In Ewa Orlowska, editor, *Essays Dedicated to the Memory of Helena Rasiowa*. Springer, Heidelberg, New York, 1998.

[Batens, 1999] D. Batens. Zero logic adding up to classical logic. *Logical Studies*, (2), 1999. `http://www.logic.ru/LogStud/02/LS2.html`.

[Batens, 2000] D. Batens. A survey of inconsistency adaptive logics. In [Batens *et al.*, 2000], pages 49–74. 2000.

[Belnap, 1977] N. D. Belnap. A useful four-valued logic. In G. Epstein and J. M. Dunn, editors, *Modern Uses of Multiple-Valued Logic*, pages 7–37. Reidel, 1977.

[Béziau, 1999] Jean-Yves Béziau. Classical negation can be expressed by one of its halves. *Logic Journal of the IGPL*, 7:145–151, 1999.

[Gentzen, 1969] Gerhard Gentzen. Investigations into logical deduction. In M. E. Szabo, editor, *The Collected Works of Gerhard Gentzen*, pages 68–131. North Holland, Amsterdam, 1969.

[Kifer and Lozinskii, 1992] M. Kifer and E. L. Lozinskii. A logic for reasoning with inconsistency. *Journal of Automated Reasoning*, 9(2):179–215, 1992.

[Kraus *et al.*, 1990] S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *AI*, 44(1-2):167–207, 1990.

[Lehmann, 1992] D. Lehmann. Plausibility logic. In *Proc. 5th Ann. Conf. of the European Assoc. for CS Logic (CSL'91)*, pages 227–241. Springer-Verlag, 1992.

[Łos and Suszko, 1958] J. Łos and R. Suszko. Remarks on sentential logics. *Indagationes Mathematicae*, 20:177–183, 1958.

[Makinson, 1994] D. Makinson. General patterns in nonmonotonic reasoning. In D. M. Gabbay, C. Hogger, and J. Robinson, editors, *Handbook of Logic in Artificial Inelligence and Logic Programming*, volume 3, pages 35–110. Oxford Science Pub., 1994.

[Priest, 1991] G. Priest. Minimally inconsistent LP. *Studia Logica*, 50:322–331, 1991.

[Reiter, 1978] R. Reiter. On closed world databases. In H. Gallair and J. Minker, editors, *Logic and Databases*, pages 55–76. Plenum Press, 1978.

[Shoham, 1987] Y. Shoham. A semantical approach to nonmonotonic logics. In M. L. Ginsberg, editor, *Readings in non-monotonic reasoning*, pages 227–249. Los-Altos, CA, 1987.