

# LP, FDE and Ambiguity

Bryson Brown  
Department of Philosophy  
University of Lethbridge  
Lethbridge, Alberta  
T1K 3M4

*Abstract: In [1] I showed that a consequence relation based on the preservation of an ambiguity-based measure of inconsistency was equivalent to the consequence relation of the logic of paradox (LP) Priest's minimal paraconsistent logic, whose semantics is based on three-valued matrices due to Kleene [2]. In this paper I take the program one step further, showing how to define two more preservationist consequence relations. The first is a modification of LP, symmetrical LP (SLP). SLP deals with trivial conclusion sets as well as trivial premise sets, recovering the elegant symmetries between premises and conclusions that characterize classical logic. The second is a preservationist semantics for LP's near relative, first degree entailment (FDE). This result raises the question, just how much of 4-valued semantics in general can be captured by appeal to preservation of ambiguity measures?*

Keywords: ambiguity, assertion, consistency, denial, entailment, paraconsistency

## 1. Assertion and denial

It has been widely held that in doing logic we have no need for a distinct notion of denial as a speech act. Frege claimed that we could express the content of any denial simply by asserting the negation of the sentence denied. Austin similarly takes the position that there is no need for a neustic with negative force, since placing the negative in the phrastic accomplishes the same end. Michael Dummett has argued that the aim of the use of declarative sentences is correct description of the world, and so all such use divides up into the successful and unsuccessful *assertion* of sentences.

But there is little reason to take these claims for granted when we come to do paraconsistent logic. Dummett's position ignores the important role that rejection of claims plays in constraining our description of the world. As C.G. Hempel argued persuasively (speaking against the more doctrinaire forms of falsificationism) there is in fact a perfect symmetry between sentences subject – relative to a view of what sentences can be observed to be true *or* false – to being conclusively falsified (including sentences of the form  $(\forall x)(Fx \rightarrow Gx)$ ), and conclusively confirmed

(including sentences of the form  $(\exists x)(Fx \wedge Gx)$ ). And of course there are sentences (such as  $(\forall x)(\exists y)Rxy$ ) that can't be either conclusively confirmed or conclusively falsified. When we go out to explore and describe our world, we constrain the descriptions we accept both by asserting some sentences and by rejecting others in response to what we find. Correct denial is as much a part of successfully describing our world as correct assertion.

Of course there is still Frege's argument to consider. But the case that Frege makes is founded on the tight links between denial and negation in classical logic. These are what make the very same descriptive commitments expressible by denying a sentence, or by asserting its negation. Of course they also make the descriptive commitments expressed by asserting a sentence identical to those expressed by denying its negation. And in fact (as G. Massey and, more recently, Van McGee, have pointed out) they actually make for a head-spinning form of Quinean indeterminacy: If we don't assume that we can behaviourally distinguish assertion from the denial of the contradictory proposition, then a dual translation of our language accounts precisely for all the same behaviour, and inferences that run from left

to right on a homophonic translation are turned around, running from right to left (i.e. denying certain conclusion sets, and inferring the disjunctive denial of a premise set).

However, more important in the context of this paper is the fact that the tight links making denial dispensable in favour of the assertion of negations are lost in some paraconsistent logics. In LP in particular, for instance, dualities linking negation, disjunction, conjunction, premises and conclusions fail. So there are reasons, at least in some paraconsistent logics, to insist from the outset on retaining the expressive power that the speech-act of denial provides.

## 2. LP symmetrized

In [3] T. Parsons suggested a compromise between gappers and glutters, between “dialetheism” and “analetheism”, that he proposed to call “agnostaletheism”. His motive lay in the powerful symmetries between the expressive power of gaps and gluts; if, somehow, a single device could be applied to achieve the expressive powers of both, then the result would be a theory able to do anything done now with the aid of gaps and/or gluts. Such a theory would also, presumably, overcome some of the expressive limits of gaps and gluts.

One way to come at the problem, beginning with LP, is to note the asymmetry of LP’s elimination of triviality in classical logic. While LP’s account of the consequence relation allows us to reject classical triviality on the left,

$$\{p, \neg p\} \models \emptyset$$

it leaves us with fully classical triviality on the right:

$$\emptyset \models \{p, \neg p\}$$

This is a straightforward consequence of the definition of the semantic consequence relation:

$$\models_{LP}: \Gamma \models_{LP} \Delta \text{ iff every assignment making all members of } \Gamma \text{ at least true also makes some member of } \Delta \text{ at least true.}$$

This definition of the consequence relation is superficially well-motivated, on analogy with the classical definition:

$$\models_C: \Gamma \models_C \Delta \text{ iff every assignment making all members of } \Gamma \text{ true also makes some member of } \Delta \text{ true.}$$

However, there is a serious flaw here, for those concerned to preserve the elegant links between assertion, denial, premises and conclusions that characterize classical logic. The flaw does not affect the classical definition, because of the simple, complementary relation classical logic imposes between assertability (= satisfiability) and its dual, deniability (= can all be made false).

LP greatly extends the range of what is assertable—in fact, LP can provide a “model” of any set of sentences at all. Unlike classical logic, it constrains the consequences of inconsistent sets because it can model them, *and* because the sets of sentences satisfied by “gratuitously” inconsistent models are always supersets of those satisfied by the minimally inconsistent models<sup>1</sup> for a given premise set. Thus in LP it is the minimally inconsistent “models” of inconsistent premise sets that determine the sets’ consequences.

But LP does not extend the range of what is deniable—for example, no matter what value is assigned to ‘p’, ‘(p ∨ ¬p)’ always receives a designated value (either true or both). If we are to recover the symmetry between what is assertable and what is deniable, we must reconsider the definition of  $\models_{LP}$ .

Priest and other dialetheists describe the third value of LP, B (usually read as “both,” and sometimes identified with the set {t,f}), as a *paradoxical* value. But in the usual presentation of LP this value is treated simply as another designated value, that is, as a value that sustains assertion (i.e. that makes assertion correct). Suppose instead that we take the paradoxicality of “both” to

<sup>1</sup> In the sense that they assign “both” to a set of sentence letters such that no valuation assigning “both” only to a proper subset of the set LP-satisfies the set.

consist precisely in the fact that it is a value that sustains both assertion and denial, that is, it is both correct to assert a sentence that has the value both, and correct to deny that sentence.

This is very much in the spirit of the dialethic approach to paraconsistency, and it allows us to say in LP that a sentence is correctly deniable if and only if its negation is correctly assertable, since in LP,

$$V(\phi) = B \text{ or } F \text{ iff } V(\neg\phi) = B \text{ or } T$$

But things look a little different when we bring this more symmetrical perspective to bear on our definition of the consequence relation. Classically,  $\models$  preserves both acceptable assertability from left to right, and acceptable deniability from right to left. That is,  $\Gamma \models_C \Delta$  iff

- i. For every satisfiable extension of  $\Gamma$ ,  $\Gamma'$ , there is an element of  $\Delta$ ,  $\delta$ , such that  $\Gamma', \delta$  is satisfiable.
- ii. For every extension of  $\Delta$ ,  $\Delta'$ , such that some valuation makes all members of  $\Delta'$  false, there is an element of  $\Gamma$ ,  $\gamma$ , such that some valuation makes all members of  $\Delta'$ ,  $\gamma$  false.

The standard practice to think of consequence relations from left to right. And in the classical case this does little or no harm. Maximal consistently deniable sets are complements of maximal consistently assertable sets, so any relation that meets i. will also, willy-nilly, satisfy ii. as well.

The standard definition of  $\models_{LP}$  seems to surrender this sort of characterization in terms of extensions. After all, any extension of  $\Gamma$  is LP-satisfiable, and so is any extension of any such extension. So clause i is empty. Of course LP takes a different tack, examining all the LP-valuations that satisfy the premise set. The work of determining the consequence relation is carried by the LP-valuations that satisfy the premise set while assigning B to a minimally sufficient set of sentence letters minimally sufficient to satisfy  $\Gamma$ , because these valuations satisfy proper subsets of the sentences satisfied by more inconsistent

valuations. But we can get a version of i back by insisting on preservation of something stronger than LP-satisfiability. What we must preserve instead is satisfiability in the minimally inconsistent LP models of  $\Gamma$ .<sup>2</sup> At the propositional level, this is measured by the set L of minimal sets of sentence letters, such that assigning only the members of each element of the set the value “both” is sufficient to allow an LP-model of  $\Gamma$  to be constructed. The LP-consequences of  $\Gamma$  are those sets  $\Delta$  such that some member of  $\Delta$  is an L-preserving extension of every L-preserving extension of  $\Gamma$ .

But of course this only gives us the first of our two clauses, the one demanding preservation of acceptable assertability from left to right in the consequence relation. The second demands preservation of acceptable *deniability* from right to left. LP simply sticks with the classical account of acceptable deniability, viz. a set is acceptably deniable iff it’s possible to assign falsehood to all its members. But if we take the paradoxical status of “both” and the symmetries linking denial and assertion seriously, then we must preserve something more than just classical deniability. We must preserve the assignment of both *or* false from right to left. And of course we must, once again, preserve this under the constraint of not making things worse, that is, we must preserve the minimally inconsistent assignments that make  $\Delta$ ’s members either both or false.

The result is SLP, a logic equivalent to LP & FDE when the conclusion set is not classically trivial, and equivalent to FDE so long as either the conclusion set or the premise set is not classically trivial. This system restores the symmetry between the right to left and the left to right directions for the consequence relation, and provides a candidate logic for Parson’s agnostaletheism.

---

<sup>2</sup> In effect, then, we treat consistency for each sentence letter as a default assumption.

### 3. Ambiguity, LP and SLP

In [1] I showed that a preservationist semantics based on ambiguous projections from inconsistent sets of sentences could be used to give an alternative semantics for the LP consequence relation. Given the 3-valued semantics for SLP above, it's clear that a similar semantics can be given for SLP. The details are straightforward. From left to right we use ambiguity to produce consistently assertable images of inconsistent sets, and then insist that our consequence relation preserve some of the minimal sets of sentence letters whose ambiguity is sufficient to project a consistent image. Symmetrically, from right to left we use ambiguity to project consistently deniable images of tautologous sets, and insist that our consequence relation preserve some of the minimal sets of sentence letters whose ambiguity is sufficient for such projections. The idea that interesting consequence relations need not be thought of as preserving *truth* is due to P.K. Schotch and R.E. Jennings- see, for example [4],[5].

Consider the results: First, as in LP, we block trivialization of inconsistent sets on the left-

$$p, \neg p \not\equiv \emptyset$$

This is because  $p, \neg p$  has consistent images that can be projected by treating  $\{p\}$  alone as ambiguous, and there is no element of  $\emptyset$  that can be added to all the extensions of  $\{p, \neg p\}$  that preserve this property, and still preserve the property. But we also block trivialization of tautologous sets on the right

$$\emptyset \not\equiv p, \neg p$$

This is because  $p, \neg p$  has consistently deniable images that can be projected by treating  $\{p\}$  alone as ambiguous, and no element of  $\emptyset$  can be added to each extension of  $\{p, \neg p\}$  that preserves this feature, while again preserving it.

This still leaves us with a triviality problem, however: though classically trivial premise sets and conclusion sets have now

been rendered non-trivial, the preservation of acceptable assertability from left to right and acceptable deniability from right to left leaves us with triviality when both the premise set and the conclusion set are classically trivial. The variable sharing requirement fails in such cases.

### 4. FDE and ambiguity

FDE is standardly treated using Dunn's 4-valued semantics, in which T and F are regarded as the classical truth values, while B and N ("neither") allow for classically unsatisfiable sets to receive designated values, and for classically unfalsifiable sets (i.e. sets whose members cannot all be assigned the value F) to receive non-designated values.[6],[7] The result is a logic whose consequence relation satisfies certain intuitions about relevance— variable sharing, in particular, holds: whenever  $\Gamma \vdash_{\text{FDE}} \alpha$ ,  $\Gamma$  and  $\alpha$  share a propositional variable.

One standard way to present this four valued semantics for FDE is as a set of rules governing the membership of a sentences' truth value, where these values are identified with the subsets of the set of classical truth values  $\{t,f\}$ :

$$t \in V(\neg A) \text{ iff } f \in V(A)$$

$$f \in V(\neg A) \text{ iff } t \in V(A)$$

$$t \in V(A \wedge B) \text{ iff } t \in V(A) \text{ and } t \in V(B)$$

$$f \in V(A \wedge B) \text{ iff } f \in V(A) \text{ or } f \in V(B)$$

$$t \in V(A \vee B) \text{ iff } t \in V(A) \text{ or } t \in V(B)$$

$$f \in V(A \vee B) \text{ iff } f \in V(A) \text{ and } f \in V(B)$$

Given an initial assignment to the atomic sentences of  $N=\emptyset$ ,  $F=\{f\}$ ,  $T=\{t\}$ ,  $B=\{t,f\}$ , these rules generate an assignment to all the sentences of the language. The consequence relation is straightforwardly defined in the usual way (once it is made clear that T and B are the designated values):

$\Gamma \vdash_{\text{FDE}} \Delta$  iff every such assignment that assigns T or B to every element of  $\Gamma$  also assigns T or B to at least one element of  $\Delta$ .

Providing an ambiguity-based semantics for this logic is fairly straightforward. We begin by connecting Dunn-valuations with projection-pairs:

**Lemma:** Given  $V, V', V'', \gamma, \mathbf{R}$ , where:

- $V$  is a Dunn-valuation  
 - $V'$  is a gap-valuation such that, for every sentence letter  $s$ :

$V(s) = T, F, N \rightarrow V'(s) = T, F, N$   
 $V(s) = B \rightarrow V'$  dispenses with  $s$ ,  
 but adds new letters  $s_T$  and  $s_F$ ,  
 where  $V'(s_T) = T$  and  $V'(s_F) = F$ .

- $V''$  is a glut-valuation such that, for every sentence letter  $s$ :

$V(s) = T, F, B \rightarrow V''(s) = T, F, B$   
 $V(s) = N \rightarrow V''$  dispenses with  $s$ ,  
 but adds new letters  $s_T$  and  $s_F$ ,  
 where  $V''(s_T) = T$  and  $V''(s_F) = F$ .

- $\gamma$  is a wff.

Let  $\mathbf{R}\gamma$  be the class of all functions

$$\mathbf{R} \in \gamma \rightarrow \gamma'$$

where  $\gamma'$  results from  $\gamma$  by replacing each instance of any  $s \in V^{-1}(B)$  (for  $V'$ ) or  $V^{-1}(N)$  (for  $V''$ ) with one or the other of  $s_T$  or  $s_F$ . Each  $\mathbf{R}$  thus creates an image of  $\gamma$  under a possible disambiguation of instances of sentence letters in  $\gamma$  assigned  $B$  ( $N$ ) by  $V$ . Then

- i.  $V(\gamma) = T$  iff  $\forall \mathbf{R} \in \mathbf{R}\gamma, V'(\mathbf{R}(\gamma)) = T$ , and  $\forall \mathbf{R} \in \mathbf{R}\gamma, V''(\mathbf{R}(\gamma)) = T$
- ii.  $V(\gamma) = F$  iff  $\forall \mathbf{R} \in \mathbf{R}\gamma, V'(\mathbf{R}(\gamma)) = F$ , and  $\forall \mathbf{R} \in \mathbf{R}\gamma, V''(\mathbf{R}(\gamma)) = F$
- iii.  $V(\gamma) = N$  iff  $\forall \mathbf{R} \in \mathbf{R}\gamma, V'(\mathbf{R}(\gamma)) = N$ , and  $\exists \mathbf{R}_1, \mathbf{R}_2 \in \mathbf{R}\gamma, V''(\mathbf{R}_1(\gamma)) = T, V'(\mathbf{R}_2(\gamma)) = F$
- iv.  $V(\gamma) = B$  iff  $\forall \mathbf{R} \in \mathbf{R}\gamma, V''(\mathbf{R}(\gamma)) = B$ , and  $\exists \mathbf{R}_1, \mathbf{R}_2 \in \mathbf{R}\gamma, V'(\mathbf{R}_1(\gamma)) = T, V'(\mathbf{R}_2(\gamma)) = F$

The proof is obtained by induction on the number of connectives in  $\gamma$ .

**Comments:**

1.  $V, \mathbf{R}\gamma$  allow us to distinguish between wffs assigned  $T, F, N$  and  $B$ .

2.  $\Gamma$  is Dunn-satisfied by  $V$  iff  $\exists \mathbf{R}_1, \dots, \mathbf{R}_n \in \mathbf{R}\gamma_1, \dots, \mathbf{R}\gamma_n$  such that  $V'(\mathbf{R}_i(\gamma_i)) = T$  for all  $\gamma_i \in \Gamma$ . Since we can regard such a collection of  $\mathbf{R}_i$  as a single function from each instance of an  $s \in V^{-1}(B)$  in  $\Gamma$  to one or another of  $s_T, s_F$ , we can say that a set  $\Gamma$  is Dunn-satisfiable iff one of its  $B$ -disambiguated images is satisfiable in a 3-valued logic including gaps. Any  $\Gamma$  that can be Dunn-satisfied by assigning  $B$  to some set of sentence letters has a classically satisfiable disambiguation which treats only those sentence letters as ambiguous.
3.  $\Delta$  is Dunn-falsified by  $V$  iff  $\exists \mathbf{R}_1, \dots, \mathbf{R}_n \in \mathbf{R}\gamma_1, \dots, \mathbf{R}\gamma_n$  such that  $V''(\mathbf{R}_i(\gamma_i)) = F$  for all  $\delta_i \in \Delta$ . As above, we can regard such a collection of  $\mathbf{R}_i$  as a single function from each instance of an  $s \in V^{-1}(N)$  in  $\Delta$  to one or another of  $s_T, s_F$ , we can say that a set  $\Delta$  is Dunn-falsifiable iff one of its  $N$ -disambiguated images is falsifiable in LP. Any  $\Gamma$  that can be Dunn-satisfied by assigning  $N$  to some set of sentence letters has a classically satisfiable disambiguation which treats only those sentence letters as ambiguous.
4. Finally, we can repeat the procedure to reduce the 3 values involved in  $V'$  and  $V''$  to the two classical values, following the same pattern. The result is a new class of projections that can stand in for the distinctions  $V'$  and  $V''$  draw with the help of  $N$  and  $B$  respectively.

**Main Theorem:**  $\Gamma \not\equiv_{FDE} \Delta$  iff there is a projection pair  $G, D$  for  $\Gamma$  and  $\Delta$  such that, for some projected images of  $\Gamma$  and  $\Delta$ ,  $\Gamma'$  and  $\Delta', \Gamma' \not\equiv \Delta'$ .

A *projection pair* for a premise set and conclusion set is a pair of sets of sentence letters  $G, D$  such that

1. Treating the letters in  $G$  ambiguously allows the projection of a classically satisfiable image of the premise set.
2. Treating the letters in  $D$  ambiguously allows the projection of a classically falsifiable image of the conclusion set
3.  $G \cap D = \emptyset$ .

1.  $\Rightarrow$ :

Suppose there is a Dunn valuation refuting  $\Gamma \models_{\text{FDE}} \Delta$ . Then there is an assignment of Dunn-values to the atomic sentences in  $\Gamma \cup \Delta$  such that all elements of  $\Gamma$  are assigned either B or T, and all elements of  $\Delta$  are assigned either N or F. But by our theorem, we can use this Dunn assignment to produce a pair of non-overlapping projection sets, one eliminating the value B and one eliminating N. Our theorem further shows there will be projections producing a satisfiable image of  $\Gamma$  based on the projection set eliminating B, and projections producing a falsifiable image of  $\Delta$  based on the projection set eliminating N. Finally, the monotonicity of B and N with respect to satisfaction/falsification ensures that these images of  $\Gamma$  and  $\Delta$  are *classically* satisfiable/falsifiable.

2.  $\Leftarrow$

Suppose we have a projection-pair,  $\langle G, D \rangle$ . Then by our theorem, assigning B to the sentence letters in  $G$  and N to the sentence letters in  $D$  will produce a Dunn-valuation satisfying  $\Gamma$  and falsifying  $\Delta$ .

## 5. Loose Ends

Thus far, our account shows only how to arrive at an ambiguity-based semantics for FDE. It does not yet provide a clear explanation in *preservationist* terms of what desirable features of premise and conclusion sets are being preserved by the consequence relation. On analogy with the accounts of LP and SLP above, I suggest that what is being preserved is a measure of the amount of ambiguity we must suppose in order to *simultaneously* project consistently assertable images of the premise set and consistently deniable images of the premise and conclusion sets. But a clear formal

expression of this measure will have to wait for another occasion.

Another important open question arises here. We have seen that the non-classical consequence relations produced with the help of extra semantic values in LP, SLP, and FDE can be captured instead by means of ambiguity. Accomplishing this for these logics is made substantially easier by the fact that their extra values (“both” and “neither”) are monotonic satisfaction (falsification) increasers. That is, changing the assignment of some sentence letter from F or T to B always increases the set of sentences satisfied on that assignment, while changing the assignment of some sentence letter from F or T to N always increases the set of sentences falsified (i.e. assigned a non-designated value) on that assignment. So, in effect, we can consider the satisfaction and falsification of images accomplished by the ambiguous projections in terms of straightforward quantification over the possible disambiguations—in the case of B, we treat any sentence having a true disambiguation as satisfied, while in the case of N we treat any sentence having a false disambiguation as falsified. But what further multi-valued logics could be captured by subtler handling of the ambiguous projections remains an open question. The resources of classical semantics combined with ambiguity projections may prove to be very rich indeed.

## 6. A Remark on Ambiguity

Ambiguity is an important phenomenon in natural language. When we try to make reasoning more explicit by translating it into formalized systems of logic, we do our best to eliminate ambiguities. But the close relations between ambiguity and logics like LP and FDE that have been proposed (among other things) to help deal with paradoxes like the Liar, suggest that in languages with sufficient expressive power, ambiguity may be inevitable, or evitable only at the cost of real inconsistency, as opposed to the *appearance* of inconsistency that ambiguity can so easily produce.

## 7. References

- [1] Bryson Brown. Yes, Virginia, there really are paraconsistent logics. *Journal of Philosophical Logic*, 28 (5): 489-500, Sept 1999.
- [2] Stephen C. Kleene. *Introduction to Metamathematics*. North Holland Publishing Co., 1967.
- [3] Terence Parsons. True contradictions. *Canadian Journal of Philosophy*. 20 (2): 335-353, September 1990.
- [4] Peter K. Schotch and R.E. Jennings. On detonating. In G. Priest, R. Routley and J. Norman, editors, *Paraconsistent Logic*: 306-327. *Philosophia Verlag*, 1989.
- [5] Raymond E. Jennings and D. Johnston. Paradox-tolerant logic. *Logique et Analyse* 26: 291-308, 1983.
- [6] J. Michael Dunn. Intuitive semantics for first-degree entailments and coupled trees. *Philosophical Studies* (29): 149-168, 1976.
- [7] Nuel D. Belnap. How a computer should think. In G. Ryle (ed.). *Contemporary Aspects of Philosophy*. Oriel Press, 1977.