Tableau Systems for Logics of Formal Inconsistency

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Abstract *The* logics of formal inconsistency (**LFI**'s) *are logics that allow to explicitly formalize the concepts of* consistency *and* inconsistency *by means of formulas of their language.* Contradictoriness*, on the other hand, can always be expressed in any logic, provided its language includes a symbol for negation. Besides being able to represent the distinction between contradiction and inconsistency,* **LFI**'s *are* non-explosive *logics, in the sense that a contradiction does not entail arbitrary statements, but yet are* gently explosive*, in the sense that, adjoining the additional requirement of consistency, then contradictoriness do cause explosion. Several logics can be seen as* **LFI**'s*, among them the great majority of paraconsistent systems developed under the Brazilian and Polish tradition. We present here tableau systems for some important* **LFI**'s*:* **bC**, **Ci** *and* **LFI1**.

Keywords: contradiction, inconsistency, consistency, paraconsistency, tableaux

1 Introduction

Although contradictoriness can be expressed in any logic, provided its language includes a symbol for negation, the concept of consistency is an essentially metalogical notion. The logics of formal inconsistency (**LFI**'s) are logics that allow to internalize the concept of consistency by means of formulas defined in their language. In formal terms, an **LFI** is any logic system that is non-explosive, in the sense that from a contradiction one is not able to derive every statement, but still is gently explosive, in the sense that, adjoining the additional requirement of consistency, then contradictoriness do cause explosion. A large class of logics can be seen as **LFI**'s, among them the great majority of paraconsistent systems developed under the Brazilian tradition (cf. [9], [7]), as well as the discussive logics developed under the Polish tradition (cf. [10]). Besides the connection to deep philosophical questions like the new definition of consistency that subsumes the model-theoretical one (which can be used, among other purposes, for a better distinction between paradoxes and antinomies) the **LFI**'s can also be applied in modeling databases (cf. [8]) and for obtaining new declarative semantics for logic programming. With such aims in mind, it is convenient to obtain suitable algorithmic presentations of **LFI**'s. We present here tableau systems for the propositional versions of the basic logic of (in)consistency, **bC**, and for its extensions **Ci** and **LFI1**. The **C**-systems are particular **LFI**'s that are obtained by (positive) conservative extensions of some previous non-paraconsistent logics, and in which the notion of consistency is expressed by a new linguistic operator. The C_n systems of da Costa (cf. [9]), for n finite, for example, are **C**-systems in which consistency can be expressed even without using the new consistency operator (in the case of C_1, for instance, the consistency of a formula A is defined by $\neg(A \wedge \neg A)$). Not all paraconsistent logics are **C**-systems: for instance, C_{min} (cf. [6]), C_ω (cf. [9]), and Pac (cf. the three-valued paraconsistent logic 'with internal implication' investigated in [1], and also in [2], under the name PI^s) are not **C**-systems. The study of **C**-systems permits to formalize, and better understand, the phenomenon of *inconsistency*, as opposed to mere *contradictoriness*. All such logics can be treated semantically by means of bivaluation semantics and possible-translations semantics, in the sense of [4] and [11].

2 bC, the basic logic of (in)-consistency

All the systems we will mention here extend the positive fragment of classical propositional logic, $\mathbf{CPL}^+$, differing from classical logic, **CPL**, only on the behavior of propositions involving negation. Consider $\wedge, \vee, \rightarrow$ and $\neg$ as our primitive connectives, and consider the addition of the following set of schematic axioms to any given axiomatization of $\mathbf{CPL}^+$, closed under Modus Ponens and Uniform Substitution:

(min1) $\vdash_{min} (A \vee \neg A)$

(min2) $\vdash_{min} (\neg\neg A \rightarrow A)$

The resulting system, C_{min}, was studied in [6]. The *basic logic of (in)consistency*, **bC**, is defined as an extension of C_{min} by the addition of a new unary connective, $\circ$, representing *consistency*, plus a new rule, realizing the Gentle Principle of Explosion:

$(bc1) \qquad \circ A, A, \neg A \vdash_{\mathbf{bC}} B.$

bC is indeed an **LFI**, i.e. a logic of formal inconsistency, and so it is in fact a **C**-system based on classical logic. Some of the most representative properties of **bC** are:

1. $\Gamma \vdash_{\mathbf{CPL}} A \Leftrightarrow \circ(\Delta), \Gamma \vdash_{\mathbf{bC}} A$, where $\circ(\Delta) = \{\circ B : B \in \Delta\}$, and Δ is a finite set of formulas;
2. $(A \wedge \neg A) \vdash_{\mathbf{bC}} \neg \circ A$;
3. $\circ A \vdash_{\mathbf{bC}} \neg(A \wedge \neg A)$;
4. $\circ A \vdash_{\mathbf{bC}} \neg(\neg A \wedge A)$.

The converses of (ii), (iii) and (iv) are not valid in **bC**, and the De Morgan laws or contraposition rules hold only in very restricted forms.

The system **bC** is sound and complete with respect to the following *bivaluation semantics*, that is, the set of all functions from the wffs of **bC** into $\{0,1\}$ such that:

$(v1)$ $v(A \wedge B) = 1 \Leftrightarrow v(A) = 1$ and $v(B) = 1$;

$(v2)$ $v(A \vee B) = 1 \Leftrightarrow v(A) = 1$ or $v(B) = 1$;

$(v3)$ $v(A \rightarrow B) = 1 \Leftrightarrow v(A) = 0$ or $v(B) = 1$;

$(v4)$ $v(\neg A) = 0 \Rightarrow v(A) = 1$;

$(v5)$ $v(\neg\neg A) = 1 \Rightarrow v(A) = 1$;

$(v6)$ $v(\circ A) = 1 \Rightarrow v(A) = 0$ or $v(\neg A) = 0$.

From $(v6)$ and $(v4)$ one obtains $(v6')$: $v(A) = 1$ and $v(\neg A) = 1 \Rightarrow v(\neg \circ A) = 1$. Based on the above mentioned semantics, one can show that the following *signed tableau rules* also constitute a sound and complete proof system for **bC**. We take for granted all usual notions related to tableaux (as, for example, that proofs are binary trees, that an a-rule means extending a node, and that a β-rule means branching a node, etc —you may consult, in that respect, [3] or [5]). The basic tableau rules are given in Tables 1 and 2.

Table 1: α-rules for **bC**

	α	α_1	α_2
1	$T(A \wedge B)$	$T(A)$	$T(B)$
2	$F(A \vee B)$	$F(A)$	$F(B)$
3	$F(A \rightarrow B)$	$T(A)$	$F(B)$
4	$T(\neg\neg A)$	$T(A)$	$T(A)$
5	$F(\neg A)$	$T(A)$	$T(A)$

A tableau branch for **bC** is closed if $F(A)$ and $T(A)$ belong to the branch. Rule 10 is a derived rule (derived tableau rules, introduced in [3], are useful but eliminable rules): If the tableau finds a node like $F(\circ A)$ it simply leaves it as it is, and does not try to break it.

Table 2: β-rules for **bC**

	β	β_1	β_2
6	$F(A \wedge B)$	$F(A)$	$F(B)$
7	$T(A \vee B)$	$T(A)$	$T(B)$
8	$T(A \to B)$	$F(A)$	$T(B)$
9	$T(\circ A)$	$F(A)$	$F(\neg A)$
10	$T(\neg A)$	$F(A)$	$F(\circ A)$

3 The logic Ci

We call **Ci** the logic obtained by addition of the following schematic rule to **bC** (defining $\bullet A$ as $\neg \circ A$ and reading $\bullet A$ as "A is inconsistent"):

(ci) $\quad \bullet A \vdash_{\mathbf{Ci}} A \wedge \neg A$

The α-rules for **Ci** are those for **bC**, plus the ones in Table 3.

Table 3: Additional rules for **Ci**

	α	α_1	α_2
11	$F(\circ A)$	$T(A)$	$T(\neg A)$
12	$T(\neg \circ A)$	$F(\circ A)$	$F(\circ A)$

The β-rules are the same as for **bC**, and the tableau closure conditions are the same. In **Ci**, $\bullet A$ and $(A \wedge \neg A)$ are obviously equivalent formulas, and **Ci** is sound and complete with respect to the bivaluation semantics given by the same clauses as in the case of **bC**, plus the converse of $(v6')$:

$(v7)$ $\; v(\neg \circ A) = 1 \Rightarrow v(A) = 1$ and $v(\neg A) = 1$.

It is noteworthy that the following rules hold in **Ci**:

1. $\circ A, \bullet A \vdash_{\mathbf{Ci}} B$;
2. $(\Gamma, B \vdash_{\mathbf{Ci}} \circ A)$ and $(\Delta, B \vdash_{\mathbf{Ci}} \bullet A)] \Rightarrow (\Gamma, \Delta \vdash_{\mathbf{Ci}} \neg B)$;
3. $\vdash_{\mathbf{Ci}} \circ \circ A$;
4. $(A \to \circ B) \vdash_{\mathbf{Ci}} (\neg \circ B \to \neg A)$;
5. $(A \to \neg \circ B) \vdash_{\mathbf{Ci}} (\circ B \to \neg A)$.

However, consistency in **Ci** is not identifiable with the negation of a contradiction, since the following do not hold in **Ci**:

1. $\neg(A \wedge \neg A) \vdash_{\mathbf{Ci}} \circ A$;
2. $\neg(\neg A \wedge A) \vdash_{\mathbf{Ci}} \circ A$.

Another interesting feature of **Ci** is that only statements about consistency or inconsistency can be provable to be consistent, that is, $\circ A$ is a theorem in **Ci** if, and only if, A is of the form $\circ B$, $\bullet B$, $\neg \circ B$ or $\neg \bullet B$, for some B.

It is also important to notice that *strong*, that is, explosive, negations can be defined inside of **bC** and of **Ci**. One such a negation, $\div$, would be obtained, for a given formula A, by defining $\div A$ as $\neg A \wedge \circ A$. But if this negation indeed has all properties of a classical negation in **Ci**, the same does not occur in **bC** (see [7]). It is possible, nevertheless, to define another strong negation, $\sim$, which behaves classically inside both **bC** and **Ci**, by defining $\sim A$ as $A \to (A \wedge (\neg A \wedge \circ A))$. Such a negation allows us to define the following mapping, for example, translating classical logic **CPL** inside of **bC** (or of **Ci**):

1. $tr(p) = p$, if p is an atomic proposition;
2. $tr(A \triangle B) = tr(A) \triangle tr(B)$, if $\triangle$ is any binary connective;
3. $tr(\neg A) = \sim tr(A)$.

The translation tr is a so-called grammatically faithful conservative translation, in the sense that it maintains the syntactical structure of the formulas. We obtain that $\Gamma \vdash_{\mathbf{CPL}} A \Leftrightarrow t(\Gamma) \vdash_{\mathbf{Ci}} t(A)$, what confers to **Ci** a very strong capability: although a sub-system of **CPL**, the system **Ci** can codify not only classical reasoning as a whole, but also metalogical properties of **CPL** concerning consistency.

4 LFI1, a maximal three-valued logic of formal inconsistency

In [8] we investigated the properties of **LFI1**, a three-valued logic of formal inconsistency apt to treat inconsistencies in databases and which also happens to be maximal relatively to classical logic, though it is in fact just a member of a much larger family of maximal three-valued **LFI**'s (see [12]). The system **LFI1** is obtained from **Ci** by adding to it the following new axioms:

(L1) $\vdash_{\mathbf{LFI1}} (A \to \neg\neg A)$

(L2) $\vdash_{\mathbf{LFI1}} (\bullet(A \wedge B) \leftrightarrow ((\bullet A \wedge B) \vee (\bullet B \wedge A)))$

(L3) $\vdash_{\mathbf{LFI1}} (\bullet(A \vee B) \leftrightarrow ((\bullet A \wedge \neg B) \vee (\bullet B \wedge \neg A)))$

(L4) $\vdash_{\mathbf{LFI1}} (\bullet(A \to B) \leftrightarrow (A \wedge \bullet B))$

The logic **LFI1** is sound and complete with respect to the three-valued matrices displayed in Table 4, where 1 and $\frac{1}{2}$ are the designated values.

Table 4: Three-valued matrices for **LFI1**

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\to$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	1	1	1

	1	$\frac{1}{2}$	0
$\neg$	0	$\frac{1}{2}$	1
$\bullet$	0	1	0

Now, it happens that **LFI1** is also sound and complete with respect to the following bivaluation semantics, defined by $(v1)-(v7)$ plus:

$(v8)$ $v(\neg\neg A) = 0 \Rightarrow v(A) = 0$;

$(v9)$ $v(\bullet(A \wedge B)) = 1 \Leftrightarrow v(\bullet A \wedge B) = 1$ or $v(\bullet B \wedge A) = 1$;

$(v10)$ $v(\bullet(A \vee B)) = 1 \Leftrightarrow v(\bullet A \wedge \neg B) = 1$ or $v(\bullet B \wedge \neg A) = 1$;

$(v11)$ $v(\bullet(A \to B)) = 1 \Leftrightarrow v(A \wedge \bullet B) = 1$.

Based on such a result, the following tableau rules can be proven to constitute a sound and complete tableau-type proof system for **LFI1**: the α- and β -rules for **LFI1** are the ones for **Ci**, plus those in Tables 5 and 6.

Table 5: α-rules for **LFI1**

	α	α_1	α_2
13	$F(\neg\neg A)$	$F(A)$	$F(A)$
14	$F(\bullet(A \wedge B))$	$F(\bullet A \wedge B)$	$F(\bullet B \wedge A)$
15	$F(\bullet(A \vee B))$	$F(\bullet A \wedge \neg B)$	$F(\bullet B \wedge \neg A)$
16	$F(\bullet(A \to B))$	$F(A \wedge \bullet B)$	$F(A \wedge \bullet B)$
17	$T(\bullet(A \to B))$	$T(A \wedge \bullet B)$	$T(A \wedge \bullet B)$

Table 6: β-rules for **LFI1**

	β	β_1	β_2
18	$T(\bullet(A \wedge B))$	$T(\bullet A \wedge B)$	$T(\bullet B \wedge A)$
19	$T(\bullet(A \vee B))$	$T(\bullet A \wedge \neg B)$	$T(\bullet B \wedge \neg A)$

Again, a tableau branch for **LFI1** is closed if either $F(A)$ and $T(A)$ belong to the branch. The rule:

	β	β_1	β_2
20	$F(\bullet A)$	$F(A)$	$F(\neg A)$

is a derived rule of type β for **LFI1**.

These tableaux proof systems show that all the above logics are decidable, and permit to treat them from the point of view of automatic theorem proving in an elegant and relatively efficient way. Since, in particular, **LFI1** is a three-valued logic, a three-signed tableau (in the sense of [3]) can also be defined. It is interesting to notice that, while our tableau system is analytic, some tableau constructions are non-well-founded trees (that is, produce loops), as it is the case of $F(\circ A)$ in **Ci** and **LFI1** (a phenomenon already noticed in [5], where similar tableaux for da Costa's C_1 have been offered). These loops, however, do not interfere with decidability, and are generated by an association of the rules (9), (10) and (11) of the tableaux. This also explains why **bC** has no tableau loops, once (11) is absent from its tableau rules.

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