Response-Ability and its Complexity

A. Polyméris, R. Contreras, M.A. Pinninghoff, E. Osses
Departamento de Ingeniería Informática y Ciencias de la Computación
Universidad de Concepción*
Concepción, Chile

Abstract *We propose a new formal model of cognitive structures and offer a first analysis of their mathematical complexity features. The structures we consider should have response-ability in all —but only in— the situations they experience. Therefore they have to be coherent, but not necessarily complete. Our understanding of knowledge is constructivistic and thus ignores some of the assumptions that characterise logical approaches. Instead, it vindicates structural economy; not only because of pragmatic reasons, but as a defining attribute of cognition. Then, mathematical complexity theory permits us to conclude that sometimes cognition is impossible. That —if $P \neq NP$— there is no efficient computational way to decide the existence of a structure that responds to an extensionally given set of experiences. Nonetheless, mathematics brings good news, when the set of experiences is already given in a comprehensive manner: There is a computational algorithm that, given any other comprehension, subexponentially decides if this second one responds to all situations the first is able to manage.*

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1 Basics

Knowledge is, in our view, a *cognitive structure* $\mathcal{G}$ that articulates a *coherent, responsible and economic* collection of *structural instances.* Lets be more precise:

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We understand that the —*perceptual*, not necessarily *real*— *world* $\mathcal{Z}$, which the cognition $\mathcal{G}$ points at, is a finite but *large* class of *situational instances*; and that $\mathcal{Z}$ is (partially) ordered according to a given, *universal comparing relation* $\sqsubseteq$. So, $(\mathcal{Z}, \sqsubseteq)$ is a poset [1]

Given a *situation* $Z \in \mathcal{Z}$, the cognitive structure $\mathcal{G}$ *responds YES*, if it has an *affirming* $V \in \mathcal{Z}$ such that $V \sqsubseteq Z$; it *responds NO*, if it holds a *negating* $W \in \mathcal{Z}$ such that $Z \sqsubseteq W$; and otherwise it is not *able to respond*. So we are only considering *categoric knowledge* that *classifies* situations into essentially two categories, according to the *structural way of being* of this *body of knowledge*. And we are assuming that $\mathcal{G} = (\mathcal{H}, \mathcal{J})$; where $\mathcal{H}, \mathcal{J} \subseteq \mathcal{Z}$ are the collections of *affirming* and *negating* structural instances, respectively; i.e. are the elements of $\mathcal{G}$ that enable its *response-ability*.

Such a cognitive structure $\mathcal{G}$ does not only *classify* or *decide* its situational answers. Its responses are not mere linguistic —*YES, NO*— utterances. Our main issue is not the *logos* —or its logical formalisation. We understand that the considered *speech-acts* can only take place when a structural commitment enables them, i.e. when the cognitive structure assumes the *responsibility* for its responses.

Therefore, since the structure is the *main issue*, we understand that the considered cognitive structures $(\mathcal{H}, \mathcal{J})$ also have to be *coherent* —not in the sense of [2], although we appreciate that author's epistemology—, i.e. they have to be such that $\forall (V, W) \in \mathcal{H} \times \mathcal{J}$, $V \not\sqsubseteq W$. This is because the members of such a pair have *opposed response faculties*: If $V \sqsubseteq W$ would be the case, then V would demand the *affirmation* of W; and at the same time, W would request the *negation* of V. So we have to assume that such *structural incoherences* would tend to *annihilate* at least one of the members of each *incoherent pair*; and thus reestablish coherence. So, because of its *coherent way of being* and the transitivity of $\sqsubseteq$, such a cognitive structure will never *contradict* itself, *respond YES and NO* in the same situation $Z \in \mathcal{Z}$.

But it may well be that $\mathcal{G}$ is not *able to respond*, neither with *YES* nor with *NO*, i.e. that the structure holds no *response-ability* for the given situation $Z \in \mathcal{Z}$ —so one could link our cognition model to three-valued logics [3]. The reflexivity of $\sqsubseteq$ only guarantees that if the situation matches a structural instance, then $\mathcal{G}$ will respond according to that coincidence. And the antisymmetry of $\sqsubseteq$ only allows to say that $\forall Z, Z' \in \mathcal{Z}$, if $Z \neq Z'$, then one can conceive a cognitive structure that *categorically distinguishes* these two situations; responding *YES* to one of the situations and *NO* to the other one.

In fact we shall not necessarily expect our cognitive structure $\mathcal{G}$ to be *complete*, i.e. to respond to *all* $Z \in \mathcal{Z}$. Such special cases will mainly have a theoretical interest —see section 3. In general, we shall only require of our *knowledge*, to prove *response-ability* in front of its *experience* $\tilde{\mathcal{Z}} \subseteq \mathcal{Z}$, i.e. a subset of situational instances, we assume to be *given*, either in an *extensive* or a *comprehensive* manner, as we shall comment in more details later —sections 2 and 3 respectively. But in any case, the *experience size* $|\tilde{\mathcal{Z}}| \in \mathcal{N}$ will typically be a *large* natural number, compared to the *structure size* that the cognition can attain without violating the *economy* we expect of any *body of knowledge*.

We do not expect of *knowledge* to be *true*, or *represent the features* of its experiences. Our epistemology is essentially *constructivistic* —somehow in the sense of [4]—, so we do not try to *seize the reality*. We expect *coherent responsibility*, as already explained. But we also understand that a structure $\mathcal{G}$ can only be considered to articulate *knowledge* if it is *economic*. I.e. if it not only is *effectively memorisable*, but also *responds efficiently*, because any given situation $Z \in \mathcal{Z}$, can *efficiently be compared* to all the structures instances.

So in fact we are asking of our cognitive structures to have the basic characteristics one usually expects of *scientific theories* that are shaped to respond to all hypothesis a certain research area recurrently presents. In such a *theoretic world*, instances correspond to *hypotheses*, and $Z \sqsubseteq Z'$ would mean that Z is *at least as general as* Z'. Then the *acceptance* of Z demands the *affirmation* of Z'; and the *rejection* of Z' requests the *negation* of Z.

However we do not expect from these *theories* to be *logic* —the difference is formally discussed in the Appendix 5.4. They are only coherent, not *potentially complete*. Because they make no use of a universal *logical negation*, but define their own *YES, NO* categories. This is a *pragmatic attitude* that naturally demands *economy*.

Here, simplifying, we shall understand that $\mathcal{G} = (\mathcal{H}, \mathcal{J})$ is *economic*, if its *size* $|\mathcal{G}| := |\mathcal{H}| + |\mathcal{J}| \in \mathcal{N}$ is a *small* number. But of course we shall only demand of our cognitive structures to be *small, compared with the large extension* of experiences they should respond to. We are interested in *ways to comprehend large classes of experiences with small cognitive structures*. Adopting the standards of mathematical complexity theory [5], we expect that the *structural sizes* of the considered cognitions, are bounded by an essentially logarithmic function of the *experience extensions* they *comprehend*.

Remember that the *size* of the usual positional cognition —notation— of a natural number is essentially a logarithm of that number.

2 Construction

If economy were not requested, *solutions* $\mathcal{G}$ would be easy to construct: Simply start with $\mathcal{G} := (\emptyset, \emptyset)$ and expose this structure to all $Z \in \tilde{\mathcal{Z}}$, one after the other: If Z is not responded by the actual, enlarged structure $\mathcal{G}$, *integrate* Z to $\mathcal{H}$ or $\mathcal{J}$, as you wish. But even if you then eliminate all *redundancy*, dismissing the structural instances that are responded by other ones —of the same *functionality*—, typically the size of $\mathcal{G}$ will grow *too much*, to be bounded by an essentially logarithmic function of the number of *experiences*. In some cases, such *non-economic* structures are easy to improve by replacing subsets of $\mathcal{H}$ or $\mathcal{J}$ by —maybe not experienced— $Z \in \mathcal{Z}$ that respond to all the structural instances of the subset they *compress*. But even then, a *comprehensive*, economic compression of $\mathcal{G}$, is normally not easy to attain:

We understand that usually the world $\mathcal{Z}$ gathers a subset of situational instances of a *universe* $(\bar{\mathcal{Z}}, \sqsubseteq)$, and that these are precisely the instances that are *not* responded by a given, small *constitutional structure* $\bar{\mathcal{G}}$. Then our interest focuses on economic, coherent, $\tilde{\mathcal{Z}}$-responsible, $\bar{\mathcal{G}}$*-complements* $\mathcal{G} = (\mathcal{H}, \mathcal{J})$. Where *complementation* requires $\mathcal{H}, \mathcal{J} \subseteq \mathcal{Z}$; i.e. a condition that can equivalently be stated in terms of $\bar{\mathcal{G}}$, in the following *natural way*: $\mathcal{G}$ has to be *coherent with the constitutional* $\bar{\mathcal{G}}$, i.e. $(\bar{\mathcal{H}}, \mathcal{J})$ and $(\mathcal{H}, \bar{\mathcal{J}})$ should be coherent; and $\mathcal{G}$ has to be *constitutionally irredundant*, i.e. $\forall V \in \mathcal{H}$, $\nexists \bar{V} \in \bar{\mathcal{H}}$ with $\bar{V} \sqsubseteq V$, and $\forall W \in \mathcal{J}$, $\nexists \bar{W} \in \bar{\mathcal{J}}$ with $W \sqsubseteq \bar{W}$.

To give another example of a *universe* where our response-ability concepts may be of interest, imagine that $\bar{\mathcal{G}}$ yields a formal model of the given, fixed *legal constitution* of a country, and that we want to *complement* this constitution with a *corpus of laws* $\mathcal{G}$ that is able to settle all legal —constitutionally not decided— questions that the country's praxis $\tilde{\mathcal{Z}}$ presents. Then, *constitutional coherence and irredundancy* condemn laws to be *wordy*.

If the order $\sqsubseteq$ of the universe $\bar{\mathcal{Z}}$ were simple —i.e. total, as would be the case if $\bar{\mathcal{Z}}$ is the set of natural numbers—, then the world $\mathcal{Z}$ would be an interval of $\bar{\mathcal{Z}}$. And since $\mathcal{Z}$ is finite, it would contain its least element $\check{Z}$ and its greatest element $\hat{Z}$. So $\mathcal{Z}$ could be *comprehended* by $(\{\check{Z}\}, \emptyset)$ or by $(\emptyset, \{\hat{Z}\})$.

But if the order is only partial, then the *world* $(\mathcal{Z}, \sqsubseteq)$ is *economically characterisable* as the subset of $\bar{\mathcal{Z}}$ that $\bar{\mathcal{G}}$ *ignores*. But this is not the *responsible* characterisation our understanding of *cognition* demands. Note that if $(\bar{\mathcal{Z}}, \sqsubseteq)$ is a lattice with meet-operator $\sqcap$ and join-operator $\sqcup$, $\mathcal{Z}$ typically is not a lattice, although it has some *similar local properties*.

Boolean lattices are *extremely partial* and at the same time widely used models —maybe even more than integer numbers. Therefore we shall concentrate on them. From now on we shall assume that E is a given finite set of boolean variables, and that the universe $\bar{\mathcal{Z}}$ is the class of all subsets of E. This means that $\bar{\mathcal{Z}}$ is a distributive lattice, and that the complement operator $\neg$ is well defined [1].

A first result proves that it is not always possible to comprehend a *large* number of experiences by a *small* cognitive structure —see the Appendix 5.1 for a *counterexample*. So in these cases, *comprehension* of $\tilde{\mathcal{Z}}$ is —in our sense— impossible.

But, given any E, $\bar{\mathcal{G}}$, $\tilde{\mathcal{Z}}$, is there at least a way to decide the *existence* of an economic and coherent $\bar{\mathcal{G}}$-complement $\mathcal{G}$ that responds to all $\tilde{\mathcal{Z}}$? If *economy* of $\mathcal{G}$ means $|\mathcal{G}| \leq \kappa$, for a given bound $\kappa \in \mathcal{N}$, then the following result proves that there is no *polynomial* algorithm able to decide this *complementation problem* —provided that, as mathematical complexity theory tends to *indicate*, the class *NP* is larger than the class P of decision problems that can be solved with a polynomially bounded computational effort.

Theorem 1:[1] The decision problem posed by the complementation issue, is *NP-complete*.

We shall not give the mathematical proof here, but develop it in the Appendix 5.5.

This result *indicates* that the *comprehension issue* —not only the *construction problem*, even the associated *existence problem*— is a very *complex one*, that it is *intractable*: If $\tilde{\mathcal{Z}}$ is given as a mere *extension of situational experiences*, then mathematical complexity theory does not know an *efficient computational way* to decide the *structural comprehensibility* issue. So we

[1]The mentioned *counterexample*, as well as this Theorem 1, are mainly due to Jean-Marie Droz, student of mathematics at the ETH of Zurich, Switzerland.

have to conclude that there is a computationally unsurmountable *complexity gap* between extension and comprehension of *the experience*.

This is worse than expected: We know that every natural number has an *economic comprehension*: its positional notation. So, if the natural number is given in an *extensive way*, to derive its *comprehension*, one would have to *count the dots* of the given extension, i.e. execute a process that clearly would demand an *effort* that can not be bounded by a polynom in the size of the comprehension.

But in our case, the needed *computing effort* cannot even be bounded by a polynom in the size of the extension —see the Appendix 5.5.

3 Comparison

But, what if $\tilde{\mathcal{Z}}$ is already given in a *comprehensive* manner? I.e. if an economic coherent $\bar{\mathcal{G}}$-complement $\tilde{\mathcal{G}}$ is given, such that $\tilde{\mathcal{Z}}$ is the subset of instances of $\bar{\mathcal{Z}}$ that are not responded by $\bar{\mathcal{G}}$, but by $\tilde{\mathcal{G}}$?

Then there may be no interest to construct another complement $\mathcal{G}$. This is unless one has reasons to be *dissatisfied* by the responses $\tilde{\mathcal{G}}$ gives and looks for an alternative structure that also responds —maybe differently— to the *problematic experiences*.

Therefore, if a second economic and coherent structure $\mathcal{G}$ is proposed, it is often of great interest to be able to decide the following *comparing question*: Does $\mathcal{G}$ also respond to all $\tilde{\mathcal{Z}}$? If so, we shall say —abbreviating— that $\mathcal{G}$ *responds to* $\tilde{\mathcal{G}}$.

Given the positional notations of two natural numbers, one may want to know if the second one is at least as large as the first. This can of course be decided with a *small computational effort* that is essentially proportional to the sum of the sizes of the two positional comprehension.

In our responsibility case, matters are not that clear. Let us first note that $\mathcal{G}$ responds to $\tilde{\mathcal{G}}$ iff $\bar{\mathcal{G}} \cup \mathcal{G}$ responds to all instances responded by $\bar{\mathcal{G}} \cup \tilde{\mathcal{G}}$. Therefore we may as well assume that $\bar{\mathcal{G}} = (\emptyset, \emptyset)$. This allows a result that reminds propositional logics, where the *logical consequence problem* can be amained to the *satisfiability problem*: We can *polynomially reduce* our *response-ability comparing problem* to the following *universality problem*: Given a coherent structure $\mathcal{G}'$, does it respond to all instances of $\bar{\mathcal{Z}}$?

Theorem 2: Given two structures $\mathcal{G}, \tilde{\mathcal{G}}$ —not necessarily coherent ones—, $\mathcal{G}$ responds to $\tilde{\mathcal{G}}$, iff: $\forall \tilde{V} \in \tilde{\mathcal{H}}$,
$(\{V \sqcap \neg\tilde{V}; V \in \mathcal{H}\}, \{W \in \mathcal{J}; \tilde{V} \sqsubseteq W\})$ is complete, and $\forall \tilde{W} \in \tilde{\mathcal{J}}$,
$(\{V \in \mathcal{H}; V \sqsubseteq \tilde{W}\}, \{W \sqcup \neg\tilde{W}; W \in \mathcal{J}\})$ is complete.
Moreover, if $\mathcal{G}$ is coherent, then all these *derived structures, generated by* $\tilde{\mathcal{G}}$, will also be coherent.
So, the considered response-ability problem is in P iff the considered universality problem is in P —for a proof, see the Appendix 5.2.

This Theorem is specially interesting, because the time-complexity of the universality problem has presented a surprise: Unlike the completeness problem —see the Appendix 5.3— and the satisfiability problem, the computational effort needed to decide the universality problem has been [6] bounded by a subexponential function of the size of $\mathcal{G}$. So *there is hope* for polynomiality, although mathematical complexity theory has not yet been able to attain that result.

4 Conclusions

At first sight, our knowledge paradigm seems to be more primitive than the logical one: it only insists on coherence and response-ability, but ignores a priori completeness requirements —see the Appendix 5.4 for a formal discussion of this difference.

Nonetheless, since it explicitly demands economy; our cognition paradigm turns out to be as demanding as the logical one, when extensive situational experience should be articulated in a comprehensive manner. But this *NP-completeness* is a property that, in our opinion, every *deep knowledge* should have, if it deserves that status because it articulates more than a mere *account*.

Nonetheless, alternative *cognitions* should be *efficiently comparable*, not with the *extensive experience*, but among them. So that communication, discussion and construction are promoted. And this is something that our model of cognition seems to allow. Unlike the classical *logical consequence problem*, that definitely seems to be intractable, our *responsibility comparison problem* could turn out to be tractable, as the result of [6] permits to hope.

In sum: The arguments that mathematical complexity theory has been developing, prove that our understanding of *response-able cognition* deserves a prolonged consideration.

5 Appendix

In this last section, we are going to develop some mathematical results to support the offered statements and prove the Theorems. Since these results assume that $\bar{\mathcal{Z}}$ is a boolean lattice, the complement $\neg$ is well defined; and, given any $\mathcal{J} \subseteq \bar{\mathcal{Z}}$, we may use the notation $\neg\mathcal{J} := \{\neg W; W \in \mathcal{J}\}$. To denote the least element and the greatest element of the boolean lattice $\bar{\mathcal{Z}}$, we will write $\bot$ and $\top$ respectively.

5.1 Counterexample

Lemma 1: [6] Let $\mathcal{G}$ be complete —not necessarily coherent— and $m \in \mathcal{N}$ be such that $\forall Y \in \mathcal{H} \cup \neg\mathcal{J}$, $m \leq |Y|$. Then $m \leq log_2(|\mathcal{G}|)$.

Proof: With $n := |E|$, $\forall Y \in \mathcal{H} \cup \neg\mathcal{J}$,
$|\{Z \in \bar{\mathcal{Z}}; Y \sqsubseteq Z\}| \leq 2^{n-m}$. Thus,
$|\mathcal{G}| \cdot 2^{n-m} \geq 2^n = |\bar{\mathcal{Z}}|$; and therefore $2^m \leq |\mathcal{G}|$.

Let us now derive the *counterexample* we promised in section 2. Given $m \in \mathcal{N}$, let $E := \{(a,b) \in A \times B\}$, where A, B are sets of m boolean variables. Thus, $n = m^2$. Let $\bar{\mathcal{G}} := (\bar{\mathcal{H}}, \bar{\mathcal{J}})$, where $\bar{\mathcal{H}} := \{V^a; a \in A\}$, $\forall a \in A$, $V^a := \{(a,b); b \in B\}$, $\neg\bar{\mathcal{J}} := \{U^b; b \in B\}$, $\forall b \in B$, $U^b := \{(a,b); a \in A\}$. Note that $\bar{\mathcal{G}}$ is coherent, that $\forall a, a' \in A$, $V^a \sqcap V^{a'} = \bot$ and $\forall b, b' \in B$, $U^b \sqcap U^{b'} = \bot$. Therefore, if a structure $\mathcal{G} := (\mathcal{H}, \mathcal{J})$ is coherent with $\bar{\mathcal{G}}$, then $\forall Y \in \mathcal{H} \cup \neg\mathcal{J}$, $m \leq |Y|$. And thus, according to Lemma 1, $|\mathcal{G}| \geq 2^m = (2^{1/2})^n$; i.e. $|\mathcal{G}|$ *grows exponentially* with n.

5.2 Proof of Theorem 2

Given $\tilde{V} \in \tilde{\mathcal{H}}$, $\mathcal{G}$ responds to all $Z \in \bar{\mathcal{Z}}$ with $\tilde{V} \sqsubseteq Z$, iff it responds to all $\{Z \sqcup \tilde{V}; Z \in \bar{\mathcal{Z}}\}$, i.e. iff for all $Z \in \bar{\mathcal{Z}}$, either $\exists V \in \mathcal{H}$ with $V \sqcap \neg\tilde{V} \sqsubseteq Z$, or $\exists W \in \mathcal{J}$ with $Z \sqsubseteq W$ and $\tilde{V} \sqsubseteq W$. The proof for the $\tilde{W} \in \tilde{\mathcal{J}}$ is similar. Finally, if $\mathcal{G}$ is coherent, given any $\tilde{V} \in \tilde{\mathcal{H}}$, $V \in \mathcal{H}$ and $W \in \mathcal{J}$ with $\tilde{V} \sqsubseteq W$, since
$V \sqcap \tilde{V} \sqsubseteq W$, $V \sqcap \neg\tilde{V} \sqsubseteq W$ would imply $V \sqsubseteq W$, a contradiction.

5.3 Satisfiability of CNF

Here and in the next subsections, we shall relate our paradigm to propositional logics, to stress similarities and differences. Some questions concerning *conjunctive normal forms* [CNF] of propositional logics can be stated equivalently in our terms.

A CNF is specified by a finite set X of boolean variables and a finite set $\mathcal{C}$ of disjunctive clauses of —*negating* or *affirming*— literals of that variables. Let E be the set of all literals. For any $x \in X$, denote by $x^-, x^+ \in E$ the *negating*, respectively *affirming literal* of the variable $x \in X$, define $D_x := \{x^-, x^+\}$, and let $\mathcal{D} := \{D_x; x \in X\}$. So, a disjunctive clause $C \in \mathcal{C}$ can be represented as an instance $C \in \bar{\mathcal{Z}}$, such that $\forall D \in \mathcal{D}$, $D \not\sqsubseteq C$.

A *valuation* is a function that assigns one of the two possible *truth values* to each variable in X. So, if we call *sub-valuations* the $Y \in \bar{\mathcal{Z}}$, such that $\forall D \in \mathcal{D}$, $D \not\sqsubseteq Y$, then the *valuations* can be represented as sub-valuation Y, such that $\forall D \in \mathcal{D}$, $D \sqcap Y \neq \bot$. A sub-valuation Y *satisfies* a clause $C \in \mathcal{C}$, iff $C \sqcap Y \neq \bot$. And it *satisfies* $\mathcal{C}$ —a given CNF—, iff it satisfies all $C \in \mathcal{C}$. Note that there exists a valuation that satisfies $\mathcal{C}$, iff there exist a sub-valuation that satisfies $\mathcal{C}$. Therefore $\mathcal{C}$ is *satisfiable*, iff, defining $\mathcal{H} := \mathcal{C}$, $\mathcal{J} := \neg\mathcal{D}$ and $\mathcal{G} := (\mathcal{H}, \mathcal{J})$, $\mathcal{G}$ is not complete. Thus, the *satisfiability problem* of propositional logics can polynomially be reduced to our *completeness problem*: given a not necessarily coherent structure $\mathcal{G}$, decide if it responds to all $\bar{\mathcal{Z}}$. This proves:

Lemma 2: This *completeness problem* is *NP-complete.*

Although note also that $(\mathcal{C}, \neg\mathcal{D})$ usually will not be coherent. So, Lemma 1 does not imply that our *universality problem* —given a structure $\mathcal{G}$, decide if it is coherent and complete— is *NP-complete.* Such a conclusion would also be very *unprobable*, considering the results of [6].

5.4 Logical Consequence

Let us now discuss the *logical consequence* issue in terms of our coherent responsibility structures. Given a CNF $(X, \mathcal{C})$, assume that $\bar{\mathcal{G}} = (\bar{\mathcal{H}}, \bar{\mathcal{J}})$ is a constitutional structure, such that $\bar{\mathcal{H}} = \mathcal{C}$ and $\neg\bar{\mathcal{J}}$ gathers a *small* subset of —*typical*— sub-valuations that satisfy $\mathcal{C}$. Then, let $Z \in \bar{\mathcal{Z}}$ be a situational clause —i.e. such that $\forall D \in \mathcal{D}$, $D \not\sqsubseteq Z$. If Z is responded by $\bar{\mathcal{H}}$ —with *YES*—, then it is clearly a logical consequence of $\mathcal{C}$; since all sub-valuations that satisfy $\mathcal{C}$, will also satisfy Z. And if $\bar{\mathcal{J}}$ responds —*NO*—, then evidently Z is *not* a logical consequence of the given CNF. So, if our constitutional structure $\bar{\mathcal{G}}$ responds, its answers will always be *logical.*

But $\bar{\mathcal{G}}$ will typically not be able to respond to all the situational clauses that may be of interest. Then, logics would aboard its standard, usually very complex process of *inferences.* Our paradigm instead, naturally envisages the construction of a coherent and economic $\bar{\mathcal{G}}$-complement $\mathcal{G}$. Before that, it may add $\mathcal{D}$ to $\bar{\mathcal{H}}$, and at the same time *expand* all sub-valuations $Y \in \neg\bar{\mathcal{J}}$ to valuations. So that they now not only satisfy $\mathcal{C}$, but also cohere with $\mathcal{D}$. This, to respond and that way dismiss all $Z \in \bar{\mathcal{Z}}$ that are not clauses, i.e to make sure that the instances that will build the envisaged $\mathcal{G}$, have the form of clauses.

Nevertheless, this will not guarantee that the constructed $V \in \mathcal{H}$ will turn out to be *consequences* of $\mathcal{C}$, as logics would expect. Because there may well exist valuations Y that satisfy all constitutional clauses of $\mathcal{C}$ but do not satisfy a constructed V. However, such valuations cannot be *structurally present*, neither in the constitutional $\neg\bar{\mathcal{J}}$, nor in its complement $\neg\mathcal{J}$, since this construction will have to cohere with V.

This *autonomy* —this limitation to *structural self-reference*— is the main feature that distinguishes our paradigm from *universal logics.* Therefore it is important to note that the *presence* in $\neg\mathcal{J}$ of a valuation Y, can also be an *uncertain* one: Because if $W \in \mathcal{J}$, then the coherence with $\mathcal{D}$ guarantees that $\forall x \in X$, either x^- or $x^+ \in \neg W$. But $\neg W$ may allow more than one *truth value* for some variables of X. Then the *presence* of such a $\neg W$ only restricts the clauses $V \in \mathcal{H}$ to be satisfiable by at least one of the *certain valuations* that $W \in \mathcal{J}$ permits.

Finally also note that such *uncertainties* diminish the response-abilities of the instances of $\mathcal{J}$. So, if $\mathcal{G}$ is to be economic, it may have to approach *certainty.* This *explains* the proposed *Theorem 1*, and permits the following:

5.5 Proof of Theorem 1

Lemma 3: Given $\mathcal{G} := (\mathcal{H}, \mathcal{J})$ and $e \in E$, if we define $\mathcal{G}' := (\mathcal{H}', \mathcal{J}')$ such that $\mathcal{H}' := \mathcal{H} \cup \{\{e\}\}$ and $\mathcal{J}' := \{W \backslash \{e\}; W \in \mathcal{J}\}$, then $|\mathcal{G}'| \leq |\mathcal{G}| + 1$, $\mathcal{G}'$ is coherent, and responds to $\mathcal{G}$.

Proof: The two first statements are evident. If $Z \in \bar{\mathcal{Z}}$ is responded by $\mathcal{G}$ but not by $\mathcal{H}$, then it is responded by $\mathcal{J}$; and if Z is not responded by $\{e\} \in \mathcal{H}'$, then $e \notin Z$, so it is responded by $\mathcal{J}'$.

Proof of Theorem 1: Given any CNF $(X, \mathcal{C})$, define E like we did above, and define $\bar{\mathcal{H}} := \{D \sqcup \{e\}; D \in \mathcal{D}, e \notin D\}$, $\bar{\mathcal{J}} := \{\perp\}$, $\tilde{\mathcal{Z}} := \mathcal{D} \cup \mathcal{C}$ and $\kappa := |X|$. Thus we get a specification of our complementation (decision) problem. Its size is $|E| \cdot (|\bar{\mathcal{G}}| + \kappa + |\tilde{\mathcal{Z}}|)$; and is therefore polynomially bounded by the size $|X| \cdot |\mathcal{C}|$ of the satisfiability problem.

Before we continue with the proof, note that Theorem 1 holds even we are assuming that the *size* of an instance of our *complementation problem* is linear in κ. We make this assumption because we understand that κ does not only specify a natural number —that can be codified by $\log_2(\kappa)$ bits— but a *register* of *size* $|E| \cdot \kappa$, able to *memorise* a potential solution $\mathcal{G}$. Also note that Theorem 1 would still hold if we would restrict the complementation problem to *normal triples* $(E, \bar{\mathcal{G}}, \kappa)$, such that $|E|$ and $|\bar{\mathcal{G}}|+\kappa$ —but not necessarily $|\tilde{\bar{\mathcal{Z}}}|$— are *polynomially related.*

Given a valuation $Y \in \bar{\mathcal{Z}}$ that satisfies $\mathcal{C}$, if we define $\mathcal{H} := \{\{e\}; e \in Y\}$ and $\mathcal{J} := \emptyset$, the structure $\mathcal{G}$ has $|\mathcal{G}| = \kappa$ elements —no one responded by $\bar{\mathcal{G}}$— and is a coherent structure that evidently responds to all $\tilde{\bar{\mathcal{Z}}}$. So, any solution of the satisfiability problem yields a solution of our complementation problem.

Thus, to prove that the satisfiability problem can polynomially be reduced to our *complementation problem* —and prove that our problem is *NP-complete*—, it suffices to prove that, whenever $\mathcal{C}$ is not *explicitly unsatisfiable* —because either $\bot \in \mathcal{C}$, or $\exists x \in X$ with $\{x^-\}, \{x^+\} \in \mathcal{C}$—, then the existence of a solution $\mathcal{G}$ of the specified complementation problem, implies the existence of a valuation that satisfies $\mathcal{C}$. This is what we prove next.

Suppose, a solution $\mathcal{G}$ exists. Then there also exists one, such that $\mathcal{H} \subseteq \{\{e\}; e \in E\}$. Because coherence of $(\mathcal{H}, \bar{\mathcal{J}})$ implies $\bot \notin \mathcal{H}$, and if $\exists V \in \mathcal{H}$ with $|V| > 1$, then, using Lemma 3, one can exchange such a V by any $\{e\}$ with $e \in V$, maintaining $|\mathcal{G}| \leq \kappa$, the coherence and the response-ability of $\mathcal{G}$. And since $\{e\}$ is not responded by $\bar{\mathcal{G}}$, the new $\mathcal{G}$ will also be a solution of our complementation problem.

Then there also exists a solution $\mathcal{G}$ that fulfils the condition of the last paragraph, and is such that $\forall x \in X$, either $\{x^-\} \in \mathcal{H}$ or $\{x^+\} \in \mathcal{H}$. Since, if this condition would not hold for a $x \in X$, then the response-ability of $\mathcal{G}$ implies the existence of a $W \in \mathcal{J}$, such that $D_x \sqsubseteq W$. Coherence of $(\bar{\mathcal{H}}, \mathcal{J})$ implies $D_x \in \mathcal{J}$. And, since we are assuming that $\mathcal{C}$ is not *explicitly unsatisfiable*, $\exists e \in D_x$, such that $\forall Z \sqsubseteq D_x \backslash \{e\}$, $Z \notin \tilde{\bar{\mathcal{Z}}}$. So choose such a e, enlarge $\mathcal{H}$ with $\{e\}$ and remove D_x from $\mathcal{J}$. Lemma 3 guarantees that the new $\mathcal{G}$ will again be a solution of our complementation problem.

Then there also exists a solution $\mathcal{G}$ that fulfils the conditions of the two last paragraphs, and is such that$\forall x \in X$, either $\{x^-\} \notin \mathcal{H}$ or $\{x^+\} \notin \mathcal{H}$. Because, since $\kappa = |X|$, otherwise there would exist a $x \in X$ that would violate the condition of the last paragraph. Then we may conclude that $\mathcal{J} = \emptyset$; and that $\mathcal{H}$ specifies a valuation $Y \sqsubseteq E$ that satisfies $\mathcal{C}$.

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