# Paraconsistency, Implication, and Truth

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**Abstract** *The present study is a sequel to [3] and [5]. We further explore preservational properties of a class of implicational connectives. The implicational approach to preservation examines what happens when the implicational connective is required to preserve various properties. We focus on a particular class of properties, called meta-valuational properties, and the class of implicational connectives required to preserve various dimensions of these properties.*

*Keywords:* Implication. Property Preservation. Finite Matrices. Paraconsistency.

## 1 A Preservationist approach

The core slogan of the preservationist approach to logic is 'find desirable properties of sentences or sets of sentences and preserve them'.

**Definition 1.1** *The connective $\to$ preserves properties of sentences. If $\to$ preserves P, and $\alpha \to \beta$ receives a designated value, then if $\alpha$ has P, $\beta$ has P.*

We focus on a particular class of properties, called *meta-valuational properties*, and the class of implicational connectives required to preserve various dimensions of these properties. Meta-valuational properties form a hierarchy of properties. The hierarchy begins with a basic property which behaves like the standard valuation of classical propositional logic. We begin by generalizing the preservationist approach to implication from [3] and [5] to an arbitrary binary connective and then evaluate a class of implicational connectives according to a criterion that we label the *Nobel measure*.

## 2 Preservationist Connectives

### 2.1 Atomic Sentences

Our approach is semantic. First, we describe the semantics of the atomic sentences. Atomic sentences, on this approach, are variables ranging over a set, $E$, the set of semantic elements. Let $a$, $b$, and $c$ range over the elements of $E$. The set of properties $\mathbf{P}_E$ is the set of binary properties distinguished by the elements of $E$. $P_i \in \mathbf{P}_E$ is represented as a function $P_i : E \to 2$. Elements of $E$ are called *profile vectors*.

**Definition 2.1** *A profile vector is an ordered list of 0's and 1's. Each place in the list represents a property. The leftmost place represents the property $P_1$ and the $i^{th}$ place, property $P_i$. 1 in the $i^{th}$ position signifies that the sentence has the property $P_i$ and 0 that the property is absent.*

The *arity* of a profile vector is $|\mathbf{P}_E|$. The set E, then, is a collection of distinct profile vectors of the same arity. We write $P_i(a) = 1(0)$ to represent the fact that the property $P_i$ is present (absent) in $a$. $D \subseteq E$, is the set of designated elements of E.

$$a \in D \Leftrightarrow P_1(a) = 1$$

We call $P_1$ the *root property*. $P_2 \ldots P_n$ are called meta-valuational properties. For the purposes of the current application, these are understood as a hierarchy of properties. Rather than being understood as properties of the sentence *per se*, they are taken as properties of the previous properties in the list. Thus, $P_4$ is a property of $P_3$ etc. [for details see [5]].

If $|E| < \omega$, then the connectives of the logical system have finite characteristic matrices. If $|E| = 2^n$ where the arity of the elements is $n$, then $E$ is said to be *complete*. Otherwise, $E$ is *incomplete*.

**Example 2.2** *Let* $\boldsymbol{P}_E = \{P_1, P_2, P_3\}$, *and* $E = \{111, 101, 100, 011, 010, 000\}$. $D = \{111, 101, 100\}$. *The arity of the profile vectors is 3. $E$ is incomplete, since* $|E| < 2^3$. *Adding* 110 *and* 001 *to $E$ extends $E$ to a complete element set,* $E^+$.

## 2.2 Binary Connectives

A binary connective $*$ for $|\mathbf{P}_E| = n$ is a collection of $n$ functions $P_1^* \dots P_n^*, P_i^* : E \times E \to 2$. $P_i^*$ is the extension function for $P_i$, and is defined as follows

$P_i^*(a * b) = 1$ iff

$(f_{11}((P_1(a)), (P_1(\beta))) = 1, \dots \& \dots$

$f_{ij}((P_i(a)), (P_j(\beta))) = 1, \dots \& \dots$

$f_{nn}((P_n(a)), (P_n(\beta))) = 1)$

where $f_{ij}$ is any *quantitative function*.

### 2.2.1 Quantitative Functions

The quantitative functions, $<$, $>$, $=$, $\leq$, $\geq$, min, max, and vac, are functions from $2 \times 2$ into 2. $<$, $>$, $=$, $\leq$, and $\geq$, are assigned 1 if the corresponding relation holds, and 0 otherwise, min and max are standard, and the vacuous function, *vac*, is defined as follows:

$$\forall x, y, vac(x, y) = 1$$

Let us call the set of quantitative functions $\mathbf{Q}$. Let $P'_i : \mathbf{P} \times \mathbf{P} \to \mathbf{Q}$. In other words, to each pair from $\mathbf{P} \times \mathbf{P}$, $P'_i$ assigns a function in $\mathbf{Q}$. We call $P'_i(P_j, P_k)$, $f_{jk}$.

**Definition 2.3** *The property profile* $P_i^*(a * b)$ *is defined as,*

$P_i^*(a * b) = 1$ *if* $\forall j, k, f_{jk}(a, b) = 1$,

*0 otherwise*

We call a property profile a *preservational profile* when $i = 1$. The rest of the profiles are collectively called a *non-alethic profile*.

The function *vac* is used to relax the requirements of property profiles. In actual practice, if for some $i, j$, $f_{ij} = vac$, we omit it from the property profile.

$f_{11}$ in the preservational profile determines whether a connective is a conjunction, disjunction, implication, or equivalence.

**Example 2.4** *Let us consider an example of a non-classical conjunction. Let our element set, $E$, be a complete collection of binary profile vectors. Let the preservational profile for conjunction be*

$P_1(a \wedge b) = 1 \Leftrightarrow min(P_1(a), P_1(b)) = 1$

*and* $P_2(a) = P_2(b)$.

*This enables us to construct the root portion of the conjunction matrix.*

| $\wedge$ | *00* | *01* | *10* | *11* |
|---|---|---|---|---|
| *00* | *0* | *0* | *0* | *0* |
| *01* | *0* | *0* | *0* | *0* |
| *10* | *0* | *0* | *1* | *0* |
| *11* | *0* | *0* | *0* | *1* |

*Now suppose we add a non-alethic profile :*

$$P_2(a \wedge b) = 1 \Leftrightarrow P_2(a) = P_2(b) = 1.$$

*Then the completed matrix is*

| $\wedge$ | *00* | *01* | *10* | *11* |
|---|---|---|---|---|
| *00* | *00* | *00* | *00* | *00* |
| *01* | *00* | *01* | *00* | *01* |
| *10* | *00* | *00* | *10* | *00* |
| *11* | *00* | *01* | *00* | *11* |

*The conjunction is nothing like classical. The conjunction could be false (non-designated) although both conjuncts are true (designated).*

## 2.3 Unary Connectives

A unary connective is a function $* : E \to E$. A trivial unary connective is one for which $\forall a \in E, *(a) = a$. Every non-trivial unary connective reverses some of the properties of some of the elements.

**Definition 2.5** *A unary connective $*$ reverses a property $P_i$ relative to set $E' \subseteq E$ iff for all $a \in E', P_i^*(a) = |P_i(a) - 1|$.*

A unary connective $*$ is *uniform*, iff for every property $P_i$ that $*$ reverses, $E' = E$. A unary connective $*$ is a *negation* iff it reverses $P_1$ and *classical* if it reverses $P_1$ only and is uniform.

# 3 Paradox-Tolerant Logic

In [3], Jennings and Johnston introduce *paradox-tolerant logic* (hereafter PTL), the first explicitly preservationist implicational logic. PTL was designed to achieve implicational paraconsistency by requiring the implication to preserve additional properties. The implication preserves truth (i.e. designation) and a property that the authors call *fixity*. To represent this addition, the logic uses a complete set of binary profile vectors as its semantic base. An entry $(a, b)$ in the implication matrix receives a designated value iff both properties are preserved, that is $P_1(a) \leq P_1(b)$ and $P_2(a) \leq P_2(b)$. The system uses matrices for conjunction and disjunction, that, as chance would have it, are isomorphic to the matrices of Heyting's intuitionist system when an extra application of Jaśkowski's Γ-function is performed [see [2]]. The negation and disjunction are classical in the above described sense, namely the only property negation reverses is $P_1$, and $P_1(a \vee b) = \max(P_1(a), P_1(b))$. The negation and disjunction are defined by the following matrices:

| $\alpha$ | $\neg\alpha$ |
|---|---|
| 00 | 10 |
| 01 | 11 |
| 10 | 00 |
| 11 | 01 |

| $\vee$ | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | 00 | 00 | 10 | 11 |
| 01 | 00 | 01 | 10 | 11 |
| 10 | 10 | 10 | 10 | 11 |
| 11 | 11 | 11 | 11 | 11 |

The other PL connectives are defined in the standard way[1] and are classical as well. In fact, the matrices are characteristic for PL. The main difference is in the implicational connective and the *falsum* constant, which are both independent of the standard PL connectives:

| $\to$ | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | 10 | 10 | 10 | 11 |
| 01 | 01 | 11 | 01 | 11 |
| 10 | 00 | 00 | 10 | 11 |
| 11 | 01 | 01 | 01 | 11 |

| $\bot$ |
|---|
| 01 |

As we have already mentioned, PTL's main aim is implicational paraconsistency and paradox-tolerance over implication. This notion of implicational paraconsistency needs some clarification. Ordinarily, a logic $L$ is said to be implicationally paraconsistent if it satisfies

$$\exists\alpha\, \exists\beta \nvdash_L (\alpha \to (\neg\alpha \to \beta)) \tag{1}$$

If this is the appropriate criterion, then PTL is implicationally paraconsistent. In fact, many additional suspicious PL implicational theorems fail in PTL. To name a few interesting ones,

$$\bot \to \alpha$$

$$\alpha \to (\alpha \vee \beta)$$

$$(\alpha \wedge \beta) \to \alpha$$

all fail. (For a more thorough list see [3]). However, as we have elsewhere noted (see [5]), higher order counterparts of some of these theorems hold in PTL. For instance, all of

$$\bot \to (\bot \to \alpha)$$

$$\neg\alpha \to (\alpha \to (\neg\alpha \to \beta))$$

$$\alpha \to (\alpha \to (\alpha \to (\alpha \vee \beta)))$$

are theorems of PTL.

This reveals that §1 is rather feeble as a criterion of implicational paraconsistency. There is room for higher standards.

**Definition 3.1** *The implication-negation fragment IN of a logic $L$ is the set of theorems of $L$ that contain no connective other than negation and implication.*

[1] The negation is distinct from Heyting's negation and such that disjunction and conjunction are interdefinable.

Then a logic L is said to be properly implicationally paraconsistent iff

$$\exists\alpha\,\exists\beta, \forall d \in IN_L,\, d \neq x \rightarrow (y \rightarrow (((\ldots \rightarrow \beta)))) \quad (2)$$

where $x$ and $y$ range over $\{\alpha, \neg\alpha\}$. Note that §2 is weaker than

$$\exists\alpha, \exists\beta, \{\alpha, \neg\alpha\} \not\vdash \beta. \quad (3)$$

§2 could hold of L without §3 holding. In the presence of *modus ponens*, the converse is false.

There is a sense, however, in which §1 and §2 are different ends of a spectrum. To explicate the similarity of the two criteria we need the notion of implicational detonation.

**Definition 3.2** *Relative to a logic L, a set* $\Sigma$ *implicationally detonates iff* $\exists d \in IN_L, d = (x \rightarrow (y \rightarrow (z \rightarrow (((\ldots \rightarrow \beta))))))$ *where* $x, y$ *and* $z$ *range over elements of* $\Sigma$*, and* $\beta$ *is an arbitrary sentence. We call the sentence* $d$ *an implicational fuse.*

This is a sense in which the implication of PTL is an improvement over the material implication, although given an inconsistent set both implications are explosive: the theorems driving explosions in PTL need deeper nestings. From the point of view of this investigation, this need for an increase in nesting is a centrally interesting feature of PTL. Furthermore, we can generalize this need for deeper nesting into a paraconsistent measure. According to this measure, the deeper one has to nest to drive an explosion from a given inconsistent set, the more paraconsistent the logic is.

It seems obvious to us that even if some contradictions are to be tolerated, it certainly need not be the case that *all* contradictions are. Even among the ones to be tolerated, if any, some are to be more tolerated than others. A natural way to capture this fact, is to impose a partial ordering on the set of available contradictions. The ordering may, for example, represent complexity of the claims involved. The simpler contradictions explode more readily than the complex and involved ones. The most complex ones, paradoxes, may be tolerated all together. The latter kind are the contradictions that no one is able to resolve in a satisfactory fashion. Some of them may be impossible to resolve, and others may simply be impossible to resolve given the present state of our knowledge.

## 4 Nobel Measure

The Nobel measure is a measure of how explosive some set of sentences is in a given logic L. The measure concerns the set $IN_L$, and provides us with the minimally nested implicational theorem required to detonate a given explosive set. If the set is non-explosive then the measure assigns it an arbitrarily high number. The measure depends on several notions, the first one of which is the notion of depth of consequence.

**Definition 4.1** *Depth of consequence is a function* $C : \Phi \rightarrow Nat$*, where* $\Phi$ *is the set of formulae of L. The function is defined recursively as follows:*

*If* $\alpha$ *is not an implication sentence,* $C(\alpha) = 0$*. If* $\alpha$ *is an implicational sentence, and* $\gamma$ *is the consequent of* $\alpha$*, then* $C(\alpha) = 1 + C(\gamma)$*. Thus,*

$C(\alpha \vee \beta) = 0$
$C(\alpha \rightarrow \beta) = 1$
$C(\alpha \rightarrow (\beta \rightarrow \gamma)) = 2$
$C(\alpha \rightarrow (\beta \rightarrow (\gamma \rightarrow \delta))) = 3$*, etc.*

**Definition 4.2** *Fuse-measure. Let* $\Sigma$ *be a set of sentences, and L the logic generated by some system S. Let* $\alpha \in IN_L$ *be* $\Sigma$*-detonating if it is a fuse for* $\Sigma$*. Then the fuse-measure is a function* $f^S_\Sigma : IN_L \rightarrow Nat, \infty$*, such that* $f^S_\Sigma(\alpha) = C(\alpha)$*, if* $\alpha$ *is* $\Sigma$*-detonating,* $\infty$ *otherwise.*

**Definition 4.3** *The Nobel measure is a function* $\boldsymbol{N}_S : \wp(\Phi) \rightarrow Nat, \infty$*.*

$$\boldsymbol{N}_S(\Sigma) = min f^S_\Sigma(\alpha).$$

**Example 4.4**
*Material conditional-negation fragment of CL. The shortest implicational fuse for* $\{\bot\}$ *is*

$$\vdash_{CL} \bot \to \beta \qquad (4)$$

*and, hence,* $\boldsymbol{N}_{CL}(\{\bot\}) = 1$.
*If the set, however, is* $\{\alpha, \neg\alpha\}$*, then the shortest implicational fuse is*

$$\vdash_{CL} \alpha \to (\neg\alpha \to \beta) \qquad (5)$$

*and, hence,* $\boldsymbol{N}_{CL}(\{\alpha, \neg\alpha\}) = 2$.

PTL is an improvement over CL with regard to the Nobel measure.
Neither §4 nor §5 are theorems of PTL.[2] Hence,

$$\mathbf{N}_{PTL}(\{\bot\}) > 1 \,\&\, \mathbf{N}_{PTL}(\{\alpha, \neg\alpha\}) > 2.$$

The aim of the present study is to construct a sequence of logics $PTL_0 \dots PTL_n \dots$ such that for any inconsistent set $\Sigma$,

$$\mathbf{N}_{PTL_0}(\Sigma) \leq \mathbf{N}_{PTL_1}(\Sigma) \leq \dots \mathbf{N}_{PTL_n}(\Sigma) \dots$$

and for $\Sigma = \{\bot\}$ or $\{\alpha, \neg\alpha\}$

$$\mathbf{N}_{PTL_0}(\Sigma) < \mathbf{N}_{PTL_1}(\Sigma) < \dots \mathbf{N}_{PTL_n}(\Sigma) \dots$$

## 5 $PTL_n$ Sequence

We want a partial ordering on the set of elements of the system. The ordering represents complexity. Thus, $\alpha < \beta$ represents the fact that $\beta$ is more complex than $\alpha$. What we mean by complexity is purposefully left somewhat vague. An easy move is to associate the complexity of a claim with the time it would take a computational device to 'process' it. We do not make this move, although our main idea is not entirely unrelated. On our (admittedly vague) account, a sentence is more complex if it requires deeper understanding of some aspect of the world in order to be processed. What this means is that, in most cases, claims the understanding of which depends upon understanding more of our theories will be more complex. Or, as it is sometimes put, the more complex claims are the ones that depend on more theory.

Each new logic in the sequence distinguishes progressively more elements by their complexity. A natural way to build the sequence of logics capturing such a progression is to use an isomorphism of the lattice ordering of classical truth functions. Each logic in the $PTL_n$ sequence is associated with the lattice defined by $\vdash_{CL}$ over the set of all $n$-ary connectives.[3] $PTL_0$ is defined over the set of all CL-constants, $PTL_1$ over the set of unary connectives, $PTL_2$ over the set of binary connectives, etc. $\alpha \leq \beta \Leftrightarrow \alpha \vdash_{CL} \beta$. The arity of profile vectors for $PTL_n$ is $2^n$.

We define negation, disjunction, implication and countably many constants for the class of lattices as follows:

**Definition 5.1** *Disjunction. If* $a \notin D$ *and* $b \notin D$ *then* $a \vee b =$
$\max\{\, c \,|\, c \leq a \,\&\, c \leq b\}$*. Let* $\boldsymbol{B} : E \to Nat$ *be a function assigning a corresponding natural number to each of the elements. For example,* $\boldsymbol{B}(000) = 0$ *and* $\boldsymbol{B}(110) = 6$*. If either* $a \in D$ *or* $b \in D$ *then* $a \vee b = \boldsymbol{B}^{-1}(\max(\boldsymbol{B}(a), \boldsymbol{B}(b)))$.

**Definition 5.2** *Negation. If* $a \in D$ *then* $\neg a = \max\{\, b \notin D \,|\, b \leq a\}$*. If* $a \notin D$ *then* $\neg a = \min\{\, b \in D \,|\, a \leq b\}$.

**Definition 5.3** *Implication. The preservational profile of implication is easily defined:*

$$P_1(a \to b) = 1 \Leftrightarrow a \leq b$$

The issues involved in defining the non-alethic profile add some complication. In [5], we assign the non-alethic profile using a fairly involved algorithm. Here, we sacrifice some of the adherence to the original sequence of logics for the sake of simplicity. The definition will change some of the places in the implicational matrix. Here is the matrix for $PTL_1$. The items marked with a star have been changed. 0 in the second place has been replaced by 1.

[2] Nor is the obvious modification of §5, $\neg\alpha \to (\alpha \to \beta)$.

[3] This approach was suggested to us by D.K. Johnston.

| → | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | **10** | **11***  | *10* | *11* |
| 01 | **01** | **11** | *01* | *11* |
| 10 | 00 | 01* | ***10*** | ***11*** |
| 11 | 01 | 01 | ***01*** | ***11*** |

It is important to note that the new sequence still provides the desired increase in the Nobel measure from one logic to the next.

Let $E/P_1$ be the restriction of the set of elements to non-designative properties, that is, the set of elements without the leftmost property. Let $\mathbf{B}[E/P_1] : E/P_1 \rightarrow Nat$ be a function assigning to each element $a \in E/P_1$ the corresponding natural number.

Let a *non-alethic quarter* of a matrix be a segment in which $P_1$ of the antecedent and the consequent are some fixed values $x$ and $y$. Then, corresponding to four possible values of $x$ and $y$—$x = 0, y = 0$; $x = 0, y = 1$; $x = 1, y = 0$; & $x = 1, y = 1$—there are four quarters. (See the above table in which the four quarters are emphasized differently.) For every logic in the sequence, the four quarters are identical in $E/P_1$. Hence, defining the non-alethic profile requires defining only one of the quarters.

Let $A$ and $C$ stand for $\mathbf{B}[E/P_1]$ of the antecedent and the consequent respectively. Let $f(A, C)$ be a function assigning a value to the implication. Then, if $C < n$,

$$A \leq C \Rightarrow f(A, C) = A$$

$$A > C \Rightarrow f(A, C) = C + 1$$

If $C = n$,

$$A < C \Rightarrow f(A, C) = A + 1$$

$$A = C \Rightarrow f(A, C) = n$$

The non-alethic profile is then assigned to implication by $\mathbf{B}^{-1}$

**Example 5.4** *Quarters for $PTL_1$ and $PTL_2$. As can be checked against the matrix above, the $PTL_1$ non-alethic quarter is*

| A\C | *0* | *1* |
|---|---|---|
| *0* | *0* | *1* |
| *1* | *1* | *1* |

*For $PTL_2$, it is*

| A\C | *0* | *1* | *2* | *3* | *4* | *5* | *6* | *7* |
|---|---|---|---|---|---|---|---|---|
| *0* | *0* | *0* | *0* | *0* | *0* | *0* | *0* | *1* |
| *1* | *1* | *1* | *1* | *1* | *1* | *1* | *1* | *2* |
| *2* | *1* | *2* | *2* | *2* | *2* | *2* | *2* | *3* |
| *3* | *1* | *2* | *3* | *3* | *3* | *3* | *3* | *4* |
| *4* | *1* | *2* | *3* | *4* | *4* | *4* | *4* | *5* |
| *5* | *1* | *2* | *3* | *4* | *5* | *5* | *5* | *6* |
| *6* | *1* | *2* | *3* | *4* | *5* | *6* | *6* | *7* |
| *7* | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *7* |

**Definition 5.5** *Constants. Let $<$ be a linear ordering of $E - D$ by $\boldsymbol{B}$. Let $m$ be the maximal element on this ordering. Let $F$ be the countable set of false constants $\{\bot, -1, -2, -3 \ldots\}$. Then, if $f$ is the interpretation function, $f(\bot) = m$, $f(-1) = m - 1$, $f(-2) = m - 2 \ldots \forall n \leq -m, f(n) = 0$.*

# 6 Syntax

A formal system of $PTL_n$ is the usual ordered triple

$$S = \langle L,\ A,\ R\rangle$$

where $L$ is a language, $A$ is a set of axioms, and $R$ is a set of rules. As we will see shortly, the only difference between various systems in the $PTL_n$ sequence lies in the set $A$.
The language $L$ of $PTL_n$ is an ordered triple

$$\langle At,\ k,\ \Phi\rangle$$

$At$ is a denumerable set of propositional variables, $k$ the denumerable set of constants $F \cup \{\neg, \vee, \rightarrow, (, )\}$ where $F = \{\bot, -1, -2, -3, \ldots\}$ is a denumerable set of false constants. The set $\Phi$ is the usual set of formulae with the addition of the clause

$$F \subseteq \Phi$$

$R$ is a pair. Its only elements are (classical) *modus ponens*, and the rule of substitution. The contents of $A$ for various degrees is determined by the strategy for the completeness proof. In general, $|A_{PTL_n}|$ is $|A_{PTL_0}| + 3n + 1$. That is, the cardinality of some set of axioms for CL, together with an axiom for each meta-valuational property for each of the three connectives. There is an additional axiom for the root property in the case of the implicational connective.

### 6.1 An Approach to Completeness

We approach completeness from a literalist point of view. Every semantic property has its syntactic representative in the guise of the above-mentioned set $F$ of false constants. A false constant, or a set of them, enables us to express the presence or absence of a property $P_i$. We can perform this kind of translation for every property and then can use the syntactic translation of property profiles as axioms of the logic. The details are straightforward. (For details see [5]).

## 7 Increasing Paraconsistency

Let $Th_{PTL_n}$ be the logic of $PTL_n$. Since for all $n$ the matrices of $PTL_{n-1}$ are embedable into the corresponding matrices of $PTL_n$, it is easily shown that $Th_{PTL_n} \subseteq Th_{PTL_{n-1}}$. It follows that for an arbitrary inconsistent set $\Sigma$,

$$\mathbf{N}_{PTL_0}(\Sigma) \leq \mathbf{N}_{PTL_1}(\Sigma) \leq \ldots \mathbf{N}_{PTL_n}(\Sigma) \ldots.$$

Showing that for $\Sigma = \{\bot\}$ or $\{\alpha, \neg\alpha\}$

$$\mathbf{N}_{PTL_0}(\Sigma) < \mathbf{N}_{PTL_1}(\Sigma) < \ldots \mathbf{N}_{PTL_n}(\Sigma) \ldots$$

is fairly straightforward. From the function that assigns the non-alethic profile to implication, it can be discerned that $\mathbf{B}(\bot \to \alpha)$ is $\mathbf{B}[E/P_1](\alpha) + 1$ if $\mathbf{B}[E/P_1](\alpha) < \mathbf{B}[E/P_1](\bot)$, and $\mathbf{B}(\neg\bot)$ otherwise. Thus, $\mathbf{N}_{PTL_n}(\bot)$ is $|E_{PTL_n} - D_{PTL_n}|$. It is an easy exercise to show that similar result holds for $\Sigma = \{\alpha, \neg\alpha\}$.

An immediate consequence is that every logic in the $PTL_n$ sequence is finitely implicationally explosive. In other words, there are finite $m$ and $l$, $m \leq l$ such that $\mathbf{N}_{PTL_n}(\{\bot\}) = m$ and $\mathbf{N}_{PTL_n}(\{\alpha, \neg\alpha\}) = l$.

### 7.1 Intersection of $PTL_n$ Logics

Now, for certain purposes it may be desirable to have a logic in which the Nobel measure for both $\{\bot\}$, and $\{\alpha, \neg\alpha\}$ is $\infty$. There are two known preservationist ways of achieving this end. The first is to introduce profile vectors of infinite arity. The second, arguably more interesting, approach is to take the intersection of all finite $PTL_n$ cases. That is,

$$PTL_\omega = \cap\{\, PTL_n \,|\, n \in Nat \,\}$$

An interesting open question in this line of research is the question of finite axiomatization for $PTL_\omega$. The logic is decidable if it is axiomatizable, for every theorem that fails, fails on some finite $PTL_n$ matrix. Since the logic is a sublogic of CL, it would also be of interest to see whether the implication-negation fragment of the logic matches some known paraconsistent sublogic of CL.

## References


[1] N.B. Belnap. *How a computer should think.* In G. Ryle (ed.) *Contemporary Aspects of Philosophy,* Oriel Press, Stocksfield, 1976.

[2] S. Jaśkowski. *Recherches sur le système de la logique intuitioniste. Congrès International de Philosophie Scientifique, part iv, 58–61,* Paris, 1935.

[3] R.E. Jennings, and D.K. Johnston. *Paradox Tolerant Logic. Logique et Analyse,* 1983.

[4] R.E. Jennings, and P.K. Schotch. *Preservation of coherence. Studia Logica,* 1984.

[5] Darko Sarenac, and R.E. Jennings. *The Preservation of Meta-Valuational Properties and the Meta-Valuational Properties of Implication.* In *Logical Consequence: Rival Approaches and New Studies in Exact Philosophy, Logic, Mathematics and Science (Vol I).* J. Woods, and B. Brown (eds.). Hermes Science Publishers, Oxford, 2000.